МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ РФ

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ВВЕДЕНИЕ В МАТЕМАТИЧЕСКИЙ АНАЛИЗ

Учебно-методическое пособие

Рекомендовано методической комиссией факультета ВМК для иностранных студентов, обучающихся в ННГУ по направлению подготовки 010300 "Фундаментальная информатика и информационные технологии"(бакалавриат).

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Пособие представляет собой сборник материалов (определений, утверждений с элементами доказательств и задач), являющихся необходимым минимумом для возможности дальнейшего освоения студентами курса математического анализа. Предназначено для учащихся факультета иностранных студентов по направлению подготовки 010300 "Фундаментальная информатика и информационные технологии".

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PRECALCULUS

Study book

Recommended by the Methodical Commission of the Faculty of Computer Science for international students, studying at the B.Sc. programme "010300Fundamental Informatics and Information Technologies".

Nizhny Novgorod
2012

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Introductory Remarks

Precalculus consists of several branch of mathematics — arithmetic, algebra, geometry, trigonometry... All of them constitute the basement for *calculus*, which, in its turn, is a very important tool of modern mathematics. Moreover, calculus is used in a lot of applied science — from biology, physics and engineering to economics and sociology. All of these subjects involve studying quantities that are growing or shrinking or moving — in other words, they are changing. Astronomers study the motions of the planets, chemists study the interaction of substances, physicists study the interactions of physical objects. Mathematics gives a universal way for describing and studying all of these processes, and one of the central concept on this way is a *function*. When you meet it in this book, take it with due attention. Good luck!

CHAPTER 1

NUMBER SYSTEMS

1. Natural and Rational Numbers

We denote the set of **natural** numbers $\{1, 2, 3, \dots\}$ by the symbol $\mathbb{N}$. The natural numbers allow us to count things, and they have the property that addition and multiplication is closed within them: that is, if we add or multiply two natural numbers, we stay within the natural numbers.

Observe that this is not true for subtraction and division, since, for example, neither $2 - 7$ nor $2 \div 7$ are natural numbers. We then say that the natural numbers enjoy **closure** within multiplication and addition.

By appending the opposite (additive inverse) of every member of $\mathbb{N}$ to $\mathbb{N}$ we obtain the set $\mathbb{Z} = \{\dots, 3, 2, 1, 0, 1, 2, 3, \dots\}$ of **integers** ($\mathbb{Z}$ for the German word *Zahlen*, meaning number).

The closure of multiplication and addition is retained by this extension and now we also have closure under subtraction.

Also, we have gained the notion of **positivity**. This last property allows us to divide the integers into **the strictly positive**, **the strictly negative** or zero, and hence introduces an **ordering** in the rational numbers by defining $a < b$ if and only if $(b - a)$ is positive.

Enter now **the rational numbers**, commonly called **fractions**, which we denote by the symbol $\mathbb{Q}$ ($\mathbb{Q}$ for quotients). They are the numbers of the form $\frac{a}{b}$ with a, b from $\mathbb{Z}$, b non equal to 0, that is, the division of two integers, with the divisor distinct from zero.

Observe that every rational number $\frac{a}{b}$ is a solution to the equation $bx - a = 0$ (with x as the unknown).

It can be shown that the rational numbers are precisely those numbers whose decimal representation either is finite (e.g., 0.123) or is periodic (e.g., 0.(123) = 0.123123123...).

Notice that every integer is a rational number, since $a/1 = a$, for any a from $\mathbb{Z}$. Upon reaching $\mathbb{Q}$ we have formed a system of numbers having closure for the four arithmetical operations of addition, subtraction, multiplication or division.

2. Fractions

All fractions have three parts: a **numerator**, a **denominator**, and a **division symbol**. In the simple fraction, the numerator and the denominator are integers. We say that $\frac{a}{b}$ or a/b is a simple fraction, where a is the numerator, b is the denominator, or / is the division symbol.

Rule 1: Division by Zero in a Simple Fraction. *The denominator of any fraction cannot have the value zero. If the denominator of a fraction is zero, the expression is not a legal fraction because it's overall value is undefined.*

This means $\frac{6}{0}$, $\frac{-3}{0}$, $\frac{5}{2-2}$ are not legal fractions. Their values are all undefined, and hence they have no meaning. Once you encounter such a fraction in a problem, stop. You cannot proceed with the problem.

Rule 2: Zero in the Numerator of Simple Fractions. *A numerator is allowed to take on the value of zero in a fraction. Any legal fraction (denominator not equal to zero) with a numerator equal to zero has an overall value of zero.*

This means $\frac{0}{6}$, $\frac{0}{-3}$, $\frac{5-5}{2}$ all have a fraction value of zero because the numerators are equal to zero.

Rule 3: One Minus Sign in Simple Fractions. *If there is one minus sign in a simple fraction, the value of the fraction will be negative.*

Example: The minus sign can be in the numerator $\frac{-3}{4}$, in the denominator $\frac{3}{-4}$ or in front of the fraction $-\frac{3}{4}$. The value of the fraction is -0.75. ◇

Exercise: *Find the value of fractions below.*

1) $\frac{20}{-4}$; 2) $\frac{0}{5}$; 3) $\frac{-8}{0}$.

Answer: 1) -4; 2) 0 (see rule 2); 3) no answer (see rule 1).

Rule 4: More Than One Minus Sign in a Simple Fractions. *If there is an even number of minus signs in a fraction, the value of the fraction is positive. If there is an odd number of minus signs in a simple fraction, the value of the fraction is negative.*

Example: Find the value of the fraction $\frac{-4}{-13}$.
Answer: The answer is $\frac{4}{13}$. ◇

Example: Find the value of the fraction $-\frac{-3}{-8}$.
Answer: The answer is $-\frac{3}{8}$. ◇

Exercise: *Find the value of fractions below.*
1) $\frac{(-4)\times(-5)}{-2}$; 2) $\frac{-3}{-5}$; 3) $\frac{-8}{3\times 5}$.

Answer: 1) -8; 2) $\frac{3}{5}$; 3) $-\frac{8}{15}$.

Rule 5: The Division Symbol in a Simple Fractions. *The Division Symbol - in a simple fraction tells the reader that the entire expression above the division symbol is the numerator and must be treated as if it were one number, and the entire expression below the division symbol is the denominator and must be treated as if it were one number.*

A fraction written as $\frac{6+10}{4-7}$ instructs the reader that the numerator is the entire expression $6 + 10$ and that denominator is the entire expression $4 - 7$. The numerator can also be written as 16 and the denominator can also be written as -3. The division symbol acts similar to a parenthesis or a bracket. Since $\frac{6+10}{4-7}$ can be written as $\frac{16}{-3} = -\frac{16}{3}$, it is a simple fraction.

Example: Simplify the fraction $\frac{7+2}{-8+1}$.
Solution: $\frac{7+2}{-8+1} = \frac{9}{-7} = -\frac{9}{7}$. ◇

Rule 6: Properties of the Number 1. *Multiplying any number by 1 does not change the value of the number. Dividing any number by 1 does not change the value of the number.*

Example: 1) $3 \times 1 = 3$; 2) $0 \times 1 = 0$; 3) $\frac{-7}{1} = -7$. ◇

Rule 7: Different Faces of the Number 1. *The number 1 can take on many forms. 4 - 3 = 1 and 10 - 9 = 1 can be used as a substitution for the number 1 because they have a value of 1. When the numerator of a fraction is equivalent to the denominator of a fraction, the value of the fraction is 1. This only works when you have a legal fraction; i.e., the denominator does not equal zero. You can substitute one of these fractions for the numer 1.*

Example: 1) $\frac{-8}{-8} = 1$; 2) $\frac{0}{0}$ is undefined (see rule 1); 3) $\frac{a}{a} = 1$ as long as $a \neq 0$. ◇

Rule 8: Any Integer Can Be Written as a Fraction.

You can express an integer as a fraction by simply dividing by 1, or you can express any integer as a fraction by simply choosing a numerator and denominator so that the overall value is equal to the integer.

Example:
1) $3 = \frac{3}{1} = \frac{6}{2}$;
2) $-7 = \frac{-7}{1} = \frac{-77}{11} = \frac{14}{-2}$;
3) $0 = \frac{0}{2} = \frac{0}{100} = \frac{0}{-5}$. ◇

Rule 9: Factoring Integers. *To factor an integer, simply break the integer down into a group of numbers whose product equals the original number. Factors are separated by multiplication signs. Note that the number 1 is the factor of every number. All factors of a number can be divided evenly into that number.*

Example:
1) Factor the number 3.
Answer: Since $3 \times 1 = 3$, the factors of 3 are 3 and 1.
2) Factor the number 10.
Answer: Since 10 can be written as $5 \times 2 \times 1$, the factors of 10 are 10, 5, 2, and 1. The number 10 can be divided by 10, 5, 2, and 1. ◇

Rule 10: Reducing Fractions. *To reduce a simple fraction, follow the following three steps:*

(1) *Factor the numerator.*
(2) *Factor the denominator.*

(3) *Find the fraction mix that equals 1.*

Example: Reduce $\frac{15}{6}$.
Solution:
1) Rewrite the fraction with the numerator and the denominator factored:

$$\frac{15}{6} = \frac{5 \times 3}{2 \times 3}$$

Note all factors in the numerator and denominator are separated by multiplication signs.
2) Find the fraction that equals 1.

$$\frac{15}{6} = \frac{5 \times 3}{2 \times 3} = \frac{5}{2} \times \frac{3}{3} = \frac{5}{2} \times 1 = \frac{5}{2}$$

We have just illustrated that $\frac{15}{6} = \frac{5}{2}$. Although the left side of the equal sign does not look identical to the right side of the equal sign, both fractions are equivalent because they have the same value. $\diamond$

Example: Reduce the fraction $\frac{120}{180}$.
Solution: $\frac{120}{180} = \frac{12 \times 10}{9 \times 10} = \frac{12}{9} \times \frac{10}{10} = \frac{12}{9} \times 1 = \frac{12}{9} = \frac{3 \times 4}{3 \times 3} = \frac{4}{3} \times \frac{3}{3} = \frac{4}{3} \times 1 = \frac{4}{3}$.
Answer: $\frac{120}{180} = \frac{4}{3}$. $\diamond$

Exercise: Reduce the fractions below.
1) $\frac{14}{-49}$; 2) $\frac{-30}{240}$; 3) $\frac{ab^2}{b^3}$.

Answer: 1) $-\frac{2}{7}$; 2) $-\frac{1}{8}$; 3)$\frac{a}{b}$.

Rule 11: Multiplication of Simple Fractions. *To multiply two simple fractions, complete the following steps*

(1) *Multiply the numerators.*
(2) *Multiply the denominators.*
(3) *Reduce the results* (*See Rule 10*).

Example: Multiply $\frac{3}{7} \times \frac{5}{6}$.
Solution: $\frac{3}{7} \times \frac{5}{6} = \frac{3 \times 5}{7 \times 6} = \frac{15}{42} = \frac{3 \times 5}{2 \times 3 \times 7} = \frac{3}{3} \times \frac{5}{2 \times 7} = 1 \times \frac{5}{14}$. $\diamond$

Rule 12: Multiplication of a Fraction and an Integer. *To multiply a whole number and a fraction, complete the following steps*

(1) *Convert the whole number to a fraction (See Rule 8).*
(2) *Multiply the numerators.*
(3) *Multiply the denominators.*
(4) *Reduce the results (See Rule 10).*

Example: $4 \times \frac{3}{8} = \frac{4}{1} \times \frac{3}{8} = \frac{4\times 3}{1\times 8} = \frac{4\times 3}{1\times 2\times 4} = \frac{4}{4} \times \frac{3}{1\times 2} = \frac{3}{2}$. ◇

Rule 13: Multiplication of Three or More Fraction. *To multiply three or more simple fractions, complete the following three steps*

(1) *Multiply the numerators.*
(2) *Multiply the denominators.*
(3) *Reduce the results (See Rule 10).*

Example: $\frac{2}{3} \times \frac{5}{6} \times \frac{7}{15} = \frac{2\times 5\times 7}{3\times 6\times 15} = \frac{170}{270} = \frac{2\times 5\times 7}{3\times 2\times 3\times 3\times 5} = \frac{2}{2} \times \frac{5}{5} \times \frac{7}{3\times 3\times 3} = 1 \times 1 \times \frac{7}{27} = \frac{7}{27}$. ◇

Rule 14: Division of Fraction. *To divide one fraction by a second frac- tion, convert the problem to multiplication and multiply the two fractions.*

(1) *Change the sign $\div$ to $\times$ and invert the fraction to the right of the sign.*
(2) *Multiply the numerators.*
(3) *Multiply the denominators.*
(4) *Reduce the results. (See Rule 10)*

Example: $\frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{1\times 4}{2\times 3} = \frac{4}{6} = \frac{2\times 2}{2\times 3} = \frac{2}{3}$. ◇

Rule 15: Division of a Fraction by an Integer. *To divide a fraction by a whole number, convert the division process to a multiplication process, and complete the following steps*

(1) *Convert the whole number to a fraction.*
(2) *Change the sign $\div$ to $\times$ and invert the fraction to the right of the sign.*
(3) *Multiply the numerators.*
(4) *Multiply the denominators.*
(5) *Reduce the results.*

Example: $\frac{1}{2} \div 2 = \frac{1}{2} \div \frac{2}{1} = \frac{1}{2} \times \frac{1}{2} = \frac{1\times1}{2\times2} = \frac{1}{4}$. ◇

Rule 16: Division of Three or More Fraction. *To divide three or more fractions, complete the following steps*

(1) *Change the signs $\div$ to $\times$ sign and invert the fractions to the right of the signs.*
(2) *Multiply the numerators.*
(3) *Multiply the denominators.*
(4) *Reduce the results.*

Example: $\frac{3}{7} \div \frac{4}{5} \div \frac{5}{6} = \frac{3}{7} \times \frac{5}{4} \times \frac{6}{5} = \frac{3\times5\times6}{7\times4\times5} = \frac{90}{140} = \frac{3\times5\times2\times3}{7\times2\times2\times5} = \frac{2}{2} \times \frac{5}{5} \times \frac{3\times3}{7\times2} = 1 \times 1 \times \frac{9}{14} = \frac{9}{14}$ ◇

Rule 17: Building of Fractions. *To build a fraction is the reverse of reducing the fraction. Instead of searching for the 1 in a fraction so that you can reduce, you insert a 1 and build.*

Example: Create a fraction with 12 in the denominator that is equivalent to the fraction $\frac{2}{3}$.

Answer: The answer is $\frac{8}{12}$.

Solution: Recall that you can multiply any number by 1 without changing the value of the number (Rule 6). Then $\frac{2}{3} = \frac{2}{3} \times 1$.

Recall that 1 has many forms, look for the form of 1 that will result in a denominator of 12. Since 3 x 4 = 12, use the fraction $\frac{4}{4}$ as 1.

Now $\frac{2}{3} \times 1$ can be written as $\frac{2}{3} \times \frac{4}{4}$. Do multiplication without the reducing of the final fraction: $\frac{2}{3} \times 1 = \frac{2}{3} \times \frac{4}{4} = \frac{2\times4}{3\times4} = \frac{8}{12}$.

We have created a fraction with a denominator equal to 12 that is equivalent to the fraction $\frac{2}{3}$. ◇

Rule 18: Addition. *To add fractions, the denominators must be equal. Complete the following steps to add two fractions.*

(1) *Build each fraction so that both denominators are equal.*
(2) *Add the numerators of the fractions.*
(3) *The denominators will be the denominator of the built-up fractions.*
(4) *Reduce the answer.*

Example: Calculate $\frac{1}{5} + \frac{3}{5}$.
Answer: The answer is $\frac{4}{5}$.
Solution: $\frac{1}{5} + \frac{3}{5} = \frac{1+3}{5} = \frac{4}{5}$.
The denominators are the same, so you can skip step 1. The denominator of the answer will be 5. Add the numerators for the numerator in the answer: $3 + 1 = 4$. The answer is $\frac{4}{5}$. This answer is already reduced, so you can skip step 4. $\diamond$

Exercise: Add the fraction below.

(1) $\frac{4}{15} + \frac{3}{15}$;
(2) $\frac{7}{10} + \frac{3}{14}$;
(3) $\frac{10}{21} + \frac{9}{24}$.

Answer:(1) $\frac{13}{15}$; (2) $\frac{32}{35}$; (3) $\frac{143}{168}$.

Rule 19: Subtraction. *To subtract, the denominators must be equal. You essentially following the same steps as in addition.*

(1) *Build each fraction so that both denominators are equal.*
(2) *Combine the numerators according to the operation of subtraction or additions.*
(3) *The denominators will be the denominator of the built-up fractions.*
(4) *Reduce the answer.*

Example: Calculate $\frac{3}{5} - \frac{1}{5}$.
Answer: The answer is $\frac{2}{5}$.
Solution: $\frac{3}{5} - \frac{1}{5} = \frac{3-1}{5} = \frac{2}{5}$.
The denominators are the same, so you can skip step 1. The denominator of the answer will be 5. Subtract the numerators for the numerator in the answer: $3 - 1 = 2$. The answer is $\frac{2}{5}$. This answer is already reduced, so you can skip step 4. $\diamond$

Exercise: Calculate:

(1) $\frac{4}{15} - \frac{3}{15}$;
(2) $\frac{7}{10} - \frac{3}{14}$;
(3) $\frac{10}{21} - \frac{9}{24}$.

Answer:(1) $\frac{1}{15}$; (2) $\frac{17}{35}$; (3) $\frac{17}{168}$.

Rule 20: Order of Operations. *Multiplication and division must be completed before addition and subtraction.*

How do you calculate $2+3\times 7$? Is the answer 35 or is the answer 23? According to the rule $2+3\times 7 = 2+21 = 23$ is the correct answer to the above question.

How do you calculate $(2+3)\times(7-3)$? Is the answer 32, 20 or is the answer 14? Follow the rule below.

Rule 21: Parenthesis. *The parenthesis instruct you to simplify the expression within the parenthesis before you proceed.*

Then $(2+3)\times(7-3) = 5\times 4 = 20$ is the correct answer to the above problem.

Rule 22: Group of Parenthesis. *If parenthesis are enclosed in other parenthesis, work from the inside out.*

Example: Calculate $3+\{7-(2+3\times 6)+2\times 5\}-7+1$.

The expression

$$(2+3\times 6)$$

is the inner most parenthesis and must be calculated first.

$$2+3\times 6 = 2+18 = 20.$$

The initial expression is now modified to

$$3+\{7-20+2\times 5\}-7+1.$$

The next parenthesis to be calculated is

$$7-20+2\times 5 = 7-20+10 = -13+10 = -3.$$

The expression is now reduced to

$$3+\{-3\}-7+1 = 0-7+1 = -6.$$

◇

Rule 24: The Division Symbol Has the Same Role as the Parenthesis. *It instructs you to treat the quantity above the numerator as if it were enclosed in a parenthesis, and to treat the quantity below the numerator as if it were enclosed in yet another parenthesis.*

Example: Calculate $\frac{2+3\times4}{4\times5-3}+\left(\frac{7-2\times5}{4+2}\right)^2$.

Solution. The expression can be written as

$$\frac{2+3\times 4}{4\times 5-3}+\left(\frac{7-2\times 5}{4+2}\right)\times\left(\frac{7-2\times 5}{4+2}\right)$$

and multiplication must be completed before addition within each parenthesis.

$\frac{2+3\times4}{4\times5-3}+\left(\frac{7-2\times5}{4+2}\right)^2=\frac{2+12}{20-3}+\left(\frac{7-10}{4+2}\right)^2=\frac{14}{17}+\left(\frac{-3}{6}\right)^2=\frac{14}{17}+\left(-\frac{1}{2}\right)^2.$

Both parenthesis have been simplified. Now perform the multiplication to yield

$$\left(-\frac{1}{2}\right)^2=\left(-\frac{1}{2}\right)\times\left(-\frac{1}{2}\right)=\frac{1}{4}.$$

The last thing to do is the addition.

$$\frac{14}{17}+\frac{1}{4}=\frac{14}{17}\times\frac{4}{4}+\frac{1}{4}\times\frac{17}{17}=\frac{56}{68}+\frac{17}{68}=\frac{73}{68}.$$

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Complex Fractions

A complex fraction is a fraction where the numerator, denominator, or both contain a fraction.

Example: $\frac{3}{\frac{1}{2}}$ is a complex fraction. The numerator is 3 and the denominator is $\frac{1}{2}$. ◇

Example: $\frac{\frac{1}{2}}{3}$ is a complex fraction. The numerator is $\frac{1}{2}$ and the denominator is 3. ◇

Example: $\frac{\frac{1}{2}}{\frac{3}{5}}$ is a complex fraction. The numerator is $\frac{1}{2}$ and the denominator is $\frac{3}{5}$. ◇

Rule 25: Manipulation with Complex Fraction. *To manipulate complex fractions, just convert them to simple fractions and follow rules 1 through 23 for simple fractions.*

To multiply, add or subtract two complex fractions, convert the fractions to simple fractions and follow the steps you use to add or subtract two simple fractions.

Example: Add the fractions $\frac{2}{3}+\frac{5}{\frac{4}{7}}$.

Solution: $\frac{2}{3}+\frac{5}{\frac{4}{7}}=\frac{2}{3}+5\div\frac{4}{7}=\frac{2}{3}+\frac{5}{1}\times\frac{7}{4}=\frac{2}{3}+\frac{5\times 7}{4}=\frac{2}{3}+\frac{35}{4}=\frac{2}{3}\times\frac{4}{4}+\frac{35}{4}\times\frac{3}{3}=\frac{8}{12}+\frac{105}{12}=\frac{8+105}{12}=\frac{113}{12}$. ◇

Rule 26: Compound Fractions. *To manipulate compound fractions, just convert them to simple fractions and follow rules 1 through 23 for simple fractions.* A compound fraction is sometimes called a mixed number. Recall that $3\frac{1}{2}$, $4\frac{2}{5}$, $-7\frac{31}{32}$ are examples of compound fractions.

Example: Convert $3\frac{1}{2}$ to a simple fraction.

Solution: $3\frac{1}{2}$ can be written as $3+\frac{1}{2}$. Write 3 as the fraction $\frac{6}{2}$. Now $3+\frac{1}{2}$ can be written as $\frac{6}{2}+\frac{1}{2}$ and we get

$$3\frac{1}{2}=3+\frac{1}{2}=\frac{6}{2}+\frac{1}{2}=\frac{7}{2}.$$

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Rule 27: Converting Simple Fraction to a Compound Fraction. *To convert a simple fraction to a compound fraction, the numerator must be larger than the denominator. Separate the whole number first.*

Example: Convert $\frac{21}{20}$ to a compound fraction.

Solution: $\frac{21}{10}=\frac{20+1}{10}=\frac{20}{10}+\frac{1}{10}=2+\frac{1}{10}=2\frac{1}{10}$. ◇

Rule 28: Manipulation with Compound Fraction. *To add, subtract , multiplay or divide two compound fractions, convert the fractions to simple fractions and follow the steps you use to for simple fractions.*

Example: Calculate $4\frac{2}{3}+7\frac{1}{9}$.

Solution: $4\frac{2}{3}+7\frac{1}{9}=4+\frac{2}{3}+7+\frac{1}{9}=11+\frac{2}{3}\times\frac{3}{3}+\frac{1}{9}=\frac{99}{9}+\frac{6}{9}+\frac{1}{9}=\frac{106}{9}$.

◇

Example: Calculate $3\frac{1}{5}\times 10\frac{1}{2}$.

Solution: $3\frac{1}{5} \times 10\frac{1}{2} = (3 + \frac{1}{5}) \times (10 + \frac{1}{2}) = (\frac{15}{5} + \frac{1}{5}) \times (\frac{20}{10} + \frac{1}{2}) = \frac{16}{5} \times \frac{21}{2} = \frac{16\times21}{5\times2} = \frac{8\times21}{5} = \frac{168}{5}$. ◇

Rule 29: Converting Decimals to Fractions. *Multiplay the decimal fraction by 1 in a form that will remove the decimal.*

Example: Convert 2.3 to a simple fraction.
Solution: $2.3 = \frac{2.3}{1} \times \frac{10}{10} = \frac{2.3\times10}{1\times10} = \frac{23}{10}$. ◇

Exercise: Convert the decimals below to a simple fraction and to a compound number.

(1) 0.00235;
(2) 4.28;
(3) 0.0000255.

Answer: (1) $\frac{47}{20000}$; (2) $4\frac{7}{25}$; (3) $\frac{51}{2000000}$.

Example: Write the infinitely repeating decimal $2.5(17) = 2.5171717\ldots$ as the quotient of two natural numbers.

Solution: The trick is to obtain multiples of x = 2.5171717. . . so that they have the same infinite tail, and then subtract these tails, cancelling them out. So observe that

$$\begin{aligned} x &= 2.5171717...; \\ 10x &= 25.171717...; \\ 100x &= 251.71717...; \\ 1000x &= 2517.1717... \end{aligned}$$

$$1000x - 10x = 990x = 2492$$

$$x = \frac{2492}{990} = \frac{(2 \times 1246)}{(2 \times 495)} = \frac{1246}{495}$$

. ◇

Rule 30: Converting Percentages to Fractions. *Recall that* $1\% = \frac{1}{100} = 0.01$. *To convert a percentage to a fraction, simply convert* 1% *to* $\frac{1}{100}$. *To convert a percentage to a decimal, simply convert* 1% *to* 0.01.

Example: Convert 2% to a fraction.

Solution: 2% can be written as $2\times1\%$ which in turn can be written as $2 \times \frac{1}{100}$. Multiply the two fractions using the multiplication rule to get $2\% = 2 \times \frac{1}{100} = \frac{2}{1} \times \frac{1}{100} = \frac{2}{100} = \frac{1}{50}$.

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2.1. Problems

Find the value of expressions below.

2.1. $\frac{5}{6} - \frac{2}{6}$.
2.2. $\frac{5}{6} + \frac{1}{2} \times \frac{2}{3}$
2.3. $\frac{24}{34} - \left(\frac{18}{34} - \frac{11}{34}\right)$.
2.4. $11\frac{8}{9} - 6\frac{7}{9}$.

Solve equations.

2.5. $3x + 5 = 3$.
2.6. $7 - 4x = 5$.
2.7. $(x + 4\frac{2}{7}) - 3\frac{6}{7} = 6$.
2.8. $9\frac{5}{12} - (7\frac{6}{12} - y) = 2\frac{3}{12}$.
2.9. $\frac{x}{6} = 16$.
2.10. $\frac{180}{y} = 60$.

Find the value of expressions below.

2.11. $\frac{9}{11} + 1\frac{6}{11}$.
2.12. $\frac{1}{16} + \frac{15}{16}$.
2.13. $3\frac{6}{7} + 1\frac{5}{7}$.
2.14. $8\frac{3}{5} + 1\frac{4}{5}$.

Calculate the value of expressions below.

2.15. $\frac{2}{5} + \frac{3}{11}$.
2.16. $\frac{1}{4} - \frac{3}{16}$.
2.17. $\frac{3}{8} + \frac{2}{3} - \frac{1}{12}$.
2.18. $\frac{3}{16} - \left(\frac{1}{4} - \frac{1}{12}\right)$.
2.19. $\frac{1}{15}\left(\frac{6}{11} + \frac{3}{22}\right)$.
2.20. $\frac{3}{8} \times \left(2\frac{11}{14} - 3\frac{1}{7}\right)$.

Find the value of expressions below.

2.21. $3\frac{2}{3} \div 1\frac{1}{9}$.
2.22. $4\frac{2}{7} \div 1\frac{1}{11}$.
2.23. $\left(1\frac{1}{3} - \left(\frac{2}{5}\right)^2 + 3\frac{1}{75}\right) \div \frac{4}{75}$
2.24. $\left(\frac{2}{9} + \left(\frac{1}{3}\right)^2 + 3\frac{1}{3}\right) \div \frac{11}{17}$.

Find the value of expressions below.

2.25. $\frac{3.6}{0.45}$.

2.26. $\frac{2\frac{2}{3}}{4\frac{1}{3}}$.

2.27. $\frac{\frac{11}{15}}{3\frac{1}{5}}$.

2.28. $\frac{3.2}{-\frac{4}{5}}$.

2.29. $\frac{1.4\times3.6\div0.2-4.2}{\frac{3}{4}\div0.2-\frac{1}{4}}$.

2.30. $\frac{3.2\times2.3\div0.4-4.4}{\frac{2}{3}\times\frac{1}{3}+\frac{5}{9}}$.

2.2. Answers

2.1. $\frac{3}{6}$. **2.2.** $\frac{7}{6}$. **2.3.** $\frac{1}{2}$. **2.4.** $4\frac{1}{9}$. **2.5.** $-\frac{2}{3}$. **2.6.** $\frac{1}{2}$. **2.7.** $5\frac{4}{7}$. **2.8.** $\frac{1}{3}$. **2.9.** 96. **2.10.** 3. **2.11.** $1\frac{4}{11}$. **2.12.** 1. **2.13.** $5\frac{4}{7}$. **2.14.** $10\frac{2}{5}$. **2.15.** $\frac{37}{55}$. **2.16.** $\frac{1}{16}$. **2.17.** $\frac{23}{24}$. **2.18.** $\frac{1}{48}$. **2.19.** $\frac{1}{22}$. **2.20.** $-\frac{3}{28}$. **2.21.** $3\frac{3}{10}$. **2.22.** $3\frac{13}{14}$. **2.23.** $78\frac{1}{2}$. **2.24.** $5\frac{2}{5}$. **2.25.** 8. **2.26.** $\frac{2}{3}$. **2.27.** $1\frac{1}{5}$. **2.28.** -4. **2.29.** $5\frac{1}{4}$. **2.30.** 18.

3. Irrational Numbers

The third kind of decimal number is one which has a non-terminating decimal expansion that does not keep repeating. Such a number is irrational, that is, it cannot be expressed as the quotient of two integers.

An example is $3.14159265\dots$.

This is the decimal expansion for the number that we ordinarily call π which is the quotient of the length of a circle to its diameter.

Other example is $\sqrt{2} = 1.41421356\dots$.

THEOREM 3.1. *There are not rational number r such that $r^2 = 2$.*

Proof: Assume the contrary. Then there are positive integers p and q such that $r = p/q$ and p and q do not have common factors. This means that $p^2/q^2 = 2$, that is $p^2 = 2q^2$. Hence the area $B = p^2$ of a square with side length p is twice the area $S = q^2$ of the square with side length q: $B = 2S$ (look at the figure. 1, a).) Since p and q do not have common factors, B and S are the smallest squares with integer side lengths such that $B = 2S$.

Now move a copy of the smaller square to the upper right hand corner of the larger square and another one to the lower left corner (fig. 1, b)). The two squares marked by A in the picture have the same area A. The square I is intersection of the two copies of the small square. Observe that the side of the square A is $p - q$ and the

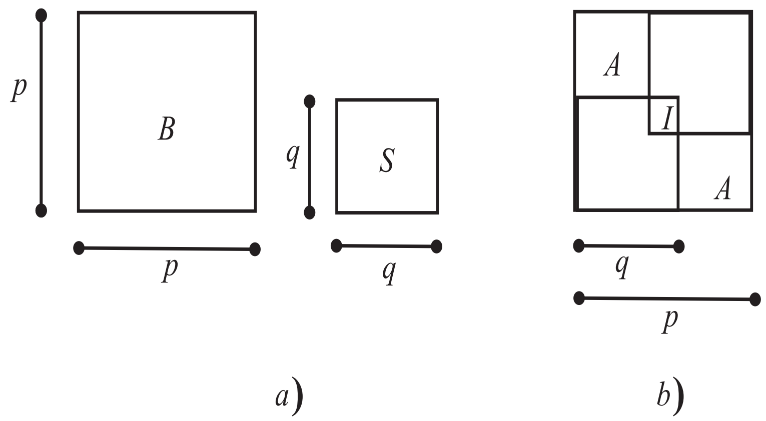


FIG. 1. To proof of the theorem 3.1

side of the square I is $2q-p$. Moreover $B = 2S+2A-I$. Since $B = 2S$ we have $I = 2A$. This is impossible, since B and S were the smallest squares with integer side lengths such that $B = 2S$. ◇

4. Real Numbers

Appending the irrational numbers to the rational numbers we obtain **the real numbers** $\mathbb{R}$.

In summary: *there are three types of real numbers*:

(i) terminating decimals,
(ii) non-terminating decimals that repeat,
(iii) non-terminating decimals that do not repeat.

Types (i) and (ii) are rational numbers $\mathbb{Q}$, type (iii) are irrational numbers $\mathbb{R} \setminus \mathbb{Q}$.

Look at the fig. 2 for classification and inclusion of subsets of the real number.

4.1. Algebraic Numbers

Observe that $\sqrt{2}$ (the square root of 2) is a solution to the equation $x^2 - 2 = 0$. A further example is $\sqrt[3]{5}$ (the cube root of 5), which is a solution to the equation $x^3 - 5 = 0$.

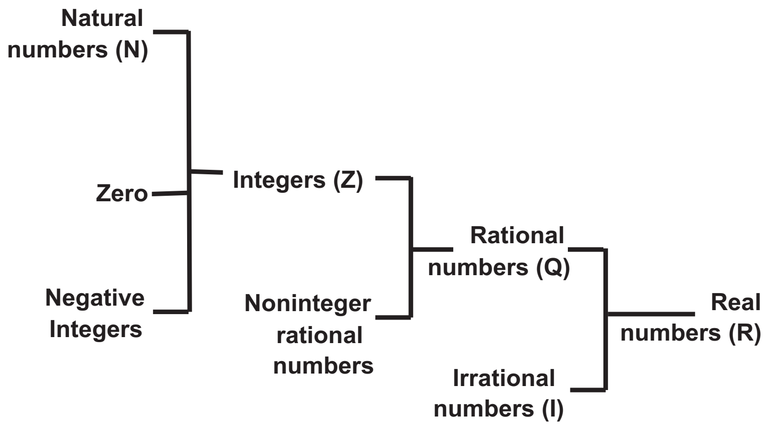


FIG. 2. Subsets of the real numbers: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

Any number which is a solution of an equation of the form $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where $a_0, a_1, \ldots, a_n$ are integers, is called an **algebraic number**.

Example: Prove that $\sqrt[3]{\sqrt{2}+1}$ is algebraic.

Solution: Work backwards: if $\sqrt[3]{\sqrt{2}+1}$, then $x^3 = \sqrt{2}+1$, which gives $(x^3-1)^2 = 2$, which is $x^6 - 2x^3 + 1 = 0$. $\diamond$

4.2. Transcendental Numbers

Are there real numbers which are not algebraic? It wasn't clear till the 19th century when it was discovered that there were irrational numbers which were not algebraic. These irrational numbers are called **transcendental numbers**.

It was later shown that numbers like π and e are transcendental. In fact, in the 19th century George Cantor proved that even though $\mathbb{N}$ and $\mathbb{R}$ are both infinite sets, their infinities are in a way "different"because they cannot be put into a one-to-one correspondence.

4.3. Geometry of the Real Numbers

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with

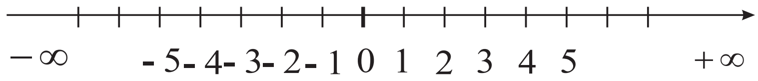


FIG. 3. Geometry of the real numbers

0 as the origin, the positive numbers increasing towards the right from 0 and the negative numbers decreasing towards the left of 0, as in the figure 3.

We append the object $+\infty$ which is larger than any real number, and the object $-\infty$ which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions:

(1) $(+\infty) + (+\infty) = +\infty$.
(2) $(-\infty) + (-\infty) = -\infty$.
(3) $x + (+\infty) = +\infty$.
(4) $x + (-\infty) = -\infty$.
(5) $x \times (+\infty) = +\infty$ if $x > 0$.
(6) $x \times (-\infty) = -\infty$ if $x > 0$.
(7) $x \times (+\infty) = -\infty$ if $x < 0$.
(8) $x \times (-\infty) = +\infty$ if $x < 0$.
(9) $\frac{x}{\pm\infty} = 0$.

4.4. Basic Real Number Properties

Let x, y, z be arbitrary elements of the set $\mathbb{R}$ of real numbers.

Addition properties.

(1) Addition is closure: $x + y$ is a unique element of $\mathbb{R}$.
(2) Addition is associative $(x + y) + z = x + (y + z)$.
(3) Addition is commutative: $x + y = y + x$.
(4) There exists an identity element 0 such that $0+x = x+0 = x$.
(5) There exists an inverse element $-x$ such that $x + (-x) = 0$.

Multiplication properties.

(1) Multiplication is closure: xy is a unique element of $\mathbb{R}$.
(2) Multiplication is associative $(xy)z = x(yz)$.
(3) Multiplication is commutative: $xy = yx$.
(4) There exists an identity element 1 such that $1x = x1 = x$.
(5) there exists an inverse element $\frac{1}{x}$ such that $x\frac{1}{x} = 1$.

Combined property – distributivity.

(1) $x(y+z) = xy + xz$; $(x+y)z = xz + yz$.

4.5. Definition of a^n

Recall that for any $a \in \mathbb{R}$ and for any $n \in \mathbb{N}$ the exponent a^n (a to the power n) is defined as a product of n factors of a:

$$a^n = \underbrace{a \times a \times \cdots \times a}_{n}.$$

a is called the base of exponent, and n is exponent or power.

If $n = 0$, $a \neq 0$ then by definition $a^0 = 1$; 0^0 is undefined.

For n a negative integer and $a \neq 0$ we define $a^n = \frac{1}{a}^{-n}$.

Using the definition one can easily verify properties of exponent listed below.

Exponent properties.

$$a^m a^n = a^{m+n}; \qquad \left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0;$$
$$(a^m)^n = a^{mn};$$
$$(ab)^m = a^m b^m; \qquad \frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}},\ a \neq 0.$$

Example: Simplify $(-x^2)^{-5} \times (-x^{-3})^4$.

Solution: $(-x^2)^{-5} \times (-x^{-3})^4 = (-1 \times x^2) - 5 \times (-1 \times x^{-3})^4 = (-1)^{-5} \times (x^2)^{-5} \times (-1)^4 \times (x^{-3})^4 = (-1) \times (x^2)^{-5} \times 1 \times (x^{-3})^4 = (-1) \times x^{2\times(-5)} \times 1 \times x^{(-3)\times 4} = -x^{-10} \times x^{-12} = -x^{-10+(-12)} = -x^{-22} = -\frac{1}{x^{22}}$. ◇

<u>**Exercise:**</u> Simplify:
(1) $(-y^7)^{-8} \times (y^{-6})^{-9}$;
(2) $(4x^{-1}y^2)^{-2}$;
(3) $(-3m^4n^{-1})^3$.

Answer: (1) $\frac{1}{y^2}$; (2) $\frac{x^2}{2y}$; (3) $-\frac{27m^{12}}{n^3}$.

Example: Calculate $\frac{5^4 15^7 + 5^{10} 27^2}{75^5}$.

Solution: Represent 15 as $15 = 3 \times 5$, then, according to the exponent properties $15^7 = (3 \times 5)^7 = 3^7 \times 5^7$ and $5^4 \times 15^7 = 5^4 \times 3^7 \times 5^7 = 5^4 \times 5^7 \times 3^7 = 5^{4+7} \times 3^7 = 5^{11} \times 3^7$.

Similarly $27^2 = (3^3)^2 = 3^{3\times 2} = 3^6$, $75^5 = (5^2\times 3)^5 = (5^2)^5 \times 3^5 = 5^{2\times 5}\times 3^5 = 5^{10}\times 3^5$. Then we have

$\frac{5^4 15^7 + 5^{10} 27^2}{75^3} = \frac{5^{11}\times 3^7 + 5^{10}\times 3^6}{5^6\times 3^3} = \frac{3^6 5^{10}(5\times 3+1)}{5^{10}\times 3^5} =$

$= 3^{6-5}5^{10-10}(5\times 3+1) = 3^1\times 5^0\times 16 = 3\times 1\times 16 = 48.$ ◇

Exercise: Simplify $\frac{27^3 15^{10} - 9^9 5^{11}}{45^9}$.

Answer: -10.

4.6. Definition of $b^{\frac{1}{n}}$

For any $b \in \mathbb{R}$ and for any $n \in \mathbb{N}$ the exponent $b^{\frac{1}{n}}$ is a number a such that $a^n = b$ and:

- If n is even and b is positive then $b^{\frac{1}{n}}$ represents the positive.
- If n is even and b is negative then $b^{\frac{1}{n}}$ does not represent any real number.
- If n is odd then there is exactly one value of $b^{\frac{1}{n}}$.
- $0^{\frac{1}{n}} = 0$.

For natural $n > 1$ and real b we define an $n-$th root radical $\sqrt[n]{b}$ by $\sqrt[n]{b} = b^{\frac{1}{n}}$. If $n = 2$ we will write $\sqrt{b}$ in place of $\sqrt[2]{b}$.

Exercise: Find the value of radical:

(1) $\sqrt[3]{64}$,
(2) $\sqrt[4]{81}$,
(3) $\sqrt[3]{-3\frac{3}{8}}$,
(4) $\sqrt[5]{0.00032}$.

Answer: (1) 4, (2) 3, (3) $-\frac{3}{2}$, (4) 0.2.

Rational Exponent Radical Conversions: For m and n positive integers ($n > 1$), and b not negative when n is even,

$$b^{\frac{m}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m.$$

Properties of radicals.

(1) $\sqrt[n]{x^n} = x$;
(2) $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$;
(3) $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$.

Exercise: Find the value of radical:

(1) $\sqrt[4]{(-3)^4}$, (3) $\sqrt[3]{(-3)^3}$,
(2) $\sqrt[5]{3^{10}}$, (4) $\sqrt[9]{27^3}$.

Answer: (1) 3, (2) 9, (3) -3, (4) 3.

4.7. Algebraic Simplification

In the following examples and problems, the term "simplify"indicates to eliminate compound fractions, factor as much as possible, reduce to a common denominator when feasible, and avoid negative exponents. It is useful to remember the following special formulas.

Special Product and Special Factoring Formulas.

$(a-b)(a+b) = a^2 - b^2$
$(a+b)^2 = a^2 + 2ab + b^2$
$(a-b)^2 = a^2 - 2ab + b^2$
$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Example: Simplify the expression $\frac{x^2-4xy+3}{(x^2-4)(x-2)}$.
Solution: $\frac{x^2-4xy+3}{x^2-4} = \frac{(x-2)^2}{(x-2)(x+2)(x-2)} = \frac{1}{x+2}$. ◇

Exercise: Simplify the expressions:
(1) $\frac{x+1}{x(x+1)^2} \times (x^2-1)$;
(2) $\frac{x+y}{x-y} \div \frac{x^2+2xy+y^2}{(x-y)^2}$.

Answer: (1) $1 - \frac{1}{x}$; (2) $\frac{x-y}{x+y}$.

4.8. Binomial Theorem

DEFINITION 4.1. *For integer $n \geq 0$, the factorial symbol $n!$ is defined by follows:*

$$0! = 1;\ 1! = 1;$$
$$n! = n(n-1)(n-2)\cdots 1,\ n \geq 2.$$

Example: $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. ◇

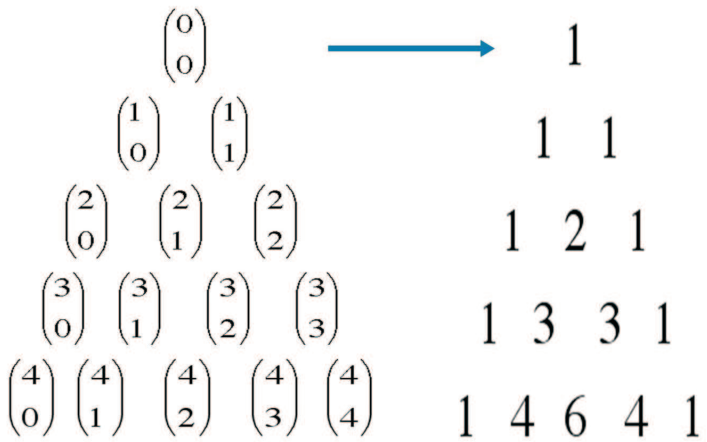


FIG. 4. Pascal triangle

DEFINITION 4.2. *For integer j, n such that $0 \le j \le n$ the symbol $\binom{n}{j}$ is defined as* $\binom{n}{j} = \frac{n!}{j!(n-j)!}$.

Example: $\binom{9}{3} = \frac{9!}{3!(9-3)!} = \frac{9!}{3!6!} = \frac{9\times8\times7\times6!}{3\times2\times1}6! = \frac{9\times8\times7}{3\times2\times1} = 84.$ $\diamond$

<u>**Exercise:**</u> Evaluate: (1) 5!; (2) 7!; (3) $\binom{7}{5}$; (4) $\binom{100}{99}$.

Answer: (1) 120; (2) 5040; (3) 21; (4) 100.

See the Pascal triangle on fig. 4 for calculating $\binom{n}{j}$.

THEOREM 4.1 (Binomial Theorem). *For any real numbers x, a and for any natural n the following formula is true:*

$$(x+a)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} a + \cdots + \binom{n}{n-1} x a^{n-1} + \binom{n}{n} a^n.$$

Example: Expand $(2y+5)^4$ using the Binomial Theorem.

Solution: $(2y+5)^4 = \sum_{j=0}^{4}\binom{4}{j}(2y)^{n-j}5^j = \binom{4}{0}(2y)^4 5^0 + \binom{4}{1}(2y))^{4-1}5^1 + \binom{4}{2}(2y)^{4-2}5^2 + \binom{4}{3}(2y)^{4-3}5^3 + \binom{4}{4}(2y)^{4-4}5^4 = 1\times 16y^4 + 4\times 8y^3\times 5 + 6\times 4y^2\times 25 + 4\times 2y\times 125 + 1\times 625 = 16y^4 + 160y^3 + 600y^2 + 1000y + 625.$ ◇

<u>Exercise:</u> Expand using the Binomial Theorem:

(1) $(a+b)^3$; (2) $(2x-3)^4$.

Answer: (1) $a^3+3a^2b+3ab^2+b^3$; (2) $16x^4-96x^3+216x^2-216x+81$.

4.9. Problems

Simplify the expression.

4.1. $2\frac{2}{3}x^2y^8(-1\frac{1}{2}xy^3)\div(3xy)^2$.

4.2. $4\frac{1}{6}a^8b^5(-1\frac{1}{5}a^5b)^3\div(6ab)^2$.

Evaluate.

4.3. $\frac{7\times 18^{3n}\times 2^{3n+1}+2\times 9^{3n}\times 4^{3n+2}}{324\times 36^{2n-2}}$.

4.4. $\frac{5\times 6^{2k+2}\times 2^{2k}-3\times 4^{2k+1}\times 3^{2k}}{42\times 12^{2k-2}}$.

Simplify the expression.

4.5. $(\sqrt{14}-2\sqrt{35})\times\frac{1}{7}\sqrt{7}+\sqrt{20}$.

4.6. $1-0.1\sqrt{5}(\sqrt{15}+\sqrt{20})$.

4.7. $(\frac{1}{3}\sqrt{39}-\frac{1}{2}\sqrt{26})\div\frac{1}{6}\sqrt{13}+\sqrt{18}$.

4.8. $(2\sqrt{38}-\sqrt{57})\times\frac{2}{19}\sqrt{19}+\sqrt{12}$.

Factorize.

4.9. $9y^2-6y+1-x^2$.

4.10. $y^2-10y+25-4m^2$.

4.11. $1000-a^3$.

4.12. $27x^3+8$.

Represent as a fraction.

4.13. $\left(\frac{m}{n} - \frac{n}{m}\right)^2 + \left(\frac{m}{n} + \frac{n}{m}\right)^2$.
4.14. $\left(\frac{a}{b} + 1\right)^2 + \left(\frac{a}{b} - 1\right)^2$.
Simplify the expressions.
4.15. $\left(\frac{x-3}{x^2-3x+9} - \frac{6x-18}{x^3+27}\right) \div \frac{5x-15}{4x^3+108}$.
4.16. $\frac{5x}{x+y} \times \left(\frac{xy+y^2}{5x^2-5xy} + xy + y^2\right) - \frac{y}{x-y}$.
4.17. $\left(\frac{a+1}{2a} + \frac{4}{a+3} - 2\right) \div \frac{a+1}{a-3} - \frac{a^2-5a+3}{2a}$.
4.18. $\left(\frac{4(x+3)}{x^2-3x} + \frac{x}{9-x^2}\right) \times \frac{x+3}{x+6} - \frac{5}{x-3}$.
4.19. $\frac{a-5}{6-3a} + \frac{4(a+1)}{a^2+4a} \div \left(\frac{9a}{a^2-16} - \frac{a+4}{a^2-4a}\right)$.

4.10. Answers

4.1.$1.5x^4y^{18}$. **4.2.**$-0.2a^{21}b^6$. **4.3.**184. **4.4.**576.

4.5.$\sqrt{2}$. **4.6.**$-\frac{sqrt3}{2}$. **4.7.**$2\sqrt{3}$. **4.8.**$4\sqrt{2}$. **4.9.**$(3y+x-1)(3y-x-1)$. **4.10.**$(y-2m-5)(y+2m-5)$. **4.11.**$(10-a)(100+10a+a^2)$. **4.12.**$(3x+2)(9x^2-18y+4)$. **4.13.**$\frac{2(m^4+n^4)}{m^2n^2}$. **4.14.**$\frac{2(a^2+b^2)}{b^2}$. **4.15.**$0.8(x-3)$. **4.16.**$5xy$. **4.17.**$\frac{2-a}{2}$. **4.18.**$-\frac{2}{x}$. **4.19.**$\frac{1}{6}$.

CHAPTER 2

EQUATIONS AND UNEQUALITIES

1. Concept of Set and Operations

A **set** is a collection of objects. We denote a set with a capital roman letter, such as A or B or C. We often will use Greek letters for denoting elements of sets and a concept of set relation presented in following two tables and diagram.

Greek Alphabets (Capital letter, small case letter, name)

A,	α,	alpha	B,	β,	beta	Γ,	γ,	gamma	Δ,	δ,	delta
E,	ε,	epsilon	Z,	ζ,	zeta	H,	η,	eta	Θ,	θ,	theta
I,	ι,	iota	K,	κ,	kappa	Λ,	λ,	lambda	M,	μ,	mu
N,	ν,	nu	Ξ,	ξ,	xi	O,	$\emptyset$,	omicron	Π,	π,	pi
P,	ρ,	rho	Σ,	σ,	sigma	T,	τ,	tau	Y,	υ,	upsilon
Φ,	φ,	phi	X,	χ,	chi	Ψ,	ψ,	psi	Ω,	ω,	omega

Example: Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9\}$.
Then $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}$, $A \cap B = \{1, 3, 5\}$, $A \setminus B = \{2, 4, 6\}$, $B \setminus A = \{7, 9\}$. $\diamond$

Example: Let $A = [-10; 2]$, $B = (-\infty; 1)$.
Then $A \cup B = (-\infty; 2]$, $A \cap B = \{[-10; 1)$, $A \setminus B = [1; 2]$, $B \setminus A = (-\infty; 10)$. $\diamond$

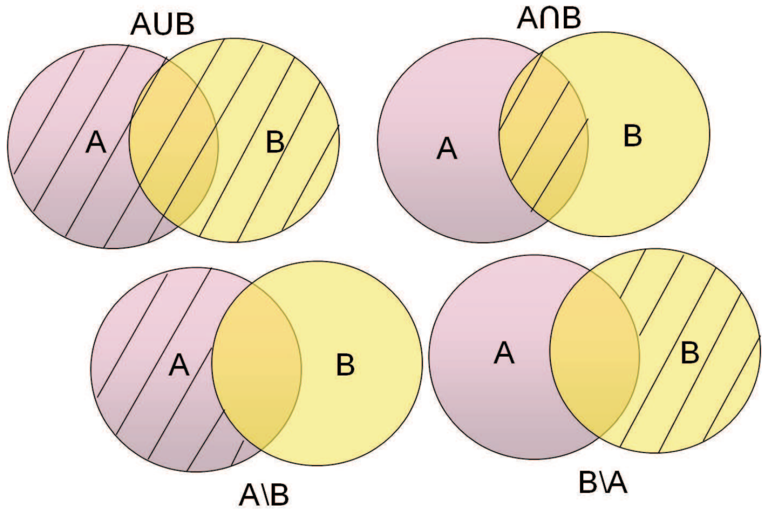


FIG. 1. Set relations

Set Relations

$\in$	Belongs to, example: $\alpha \in \{1, 2, \alpha, \beta, a, b\}$.
$\notin$	Does not belongs to, example: $\gamma \notin \{1, 2, \alpha, \beta, a, b\}$.
$\subset$	Subset symbol: $A \subset B$ means that all elements of the set A belongs to set B.
$\emptyset$	Empty set, i.e. a set that does not contain any elements.
$\cup$	Union of two sets. The union $A \cup B$ is a set containing all elements from sets A and B.
$\cap$	Intersection of two sets. The intersection $A \cap B$ is a set containing all elements that belong to both sets A and B.
$\setminus$	Difference of two sets. The difference $A \setminus B$ (A setminus B) is a set containing all elements that belong to the set A and do not belong to the set B.

Following table lists symbols and quantifiers that are usually used for abbreviate notation.

Symbols and Quantifiers

$\Rightarrow$	Follows. $S \Rightarrow T$ means that if the statements S is true then also the statement T is true. Example: $n \in \mathbb{N} \Rightarrow n^2 + n$ is even.
$\Leftrightarrow$	Equivalence of statements. $S \Leftrightarrow T$ means that statements S and T are either both true or both false.
$\forall$	For all. Example: $\forall x \in \mathbb{R} : \; x^2 \geq 0$.
$\exists$	There is. Example: $\exists n \in \mathbb{N} : \; n^2 - 6n < 0$.
$\exists!$	There is a unique. Example: $\exists! n \in \mathbb{N} : \; n^2 - 2n < 0$.
$\wedge$	And.
$\vee$	Or.
$\sum$	Greek letter Sigma, symbol for summation. Example: $\sum\limits_{i=1}^{m} i^2 = 1 + 4 + 9 + \cdots + m^2$.
(a, b)	Open interval from a to b: $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$.
$[a, b]$	Closed interval from a to b: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$.

Example: The following statements are true mathematical statements illustrating typical usage of the above symbols.

- $\forall n \in \mathbb{N} : n^2 - n$ is even.
- $\forall x > 0 : sin(x) < x$.
- $\sum\limits_{n=1}^{m} n = \frac{m(m+1)}{2}$.
- $\sum\limits_{n=1}^{m} n^2 = \frac{m^3}{3} + \frac{n^2}{2} + \frac{n}{6}$.

$\diamond$

Exercise: Using mathematical notation, write the following statements without using words:

(1) For all negative numbers x, x is smaller than $sin(x)$.
(2) For any positive real number δ, there is a positive rational number q such that δ smaller than q.

Answer:

(1) $\forall x < 0 : x < sin(x)$.
(2) $\forall \delta \in \mathbb{R}, \; r > 0 \; \exists q \in \mathbb{Q}, q > 0$: $\delta < q$.

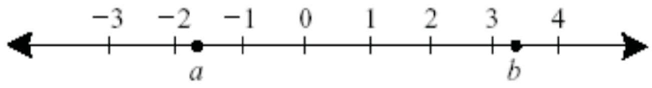


FIG. 2. Representation of real numbers

Example: Are the following statements true or false:
(1) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{Z}:\ 0 < \frac{x}{y} < 1$.
(2) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{Z}:\ 0 < \frac{x}{y} < 1$.
(3) $\forall \varepsilon > 0\ \exists \delta > 0 : |x-1| < \delta \Rightarrow |2x-2| < \varepsilon$.
(4) $\exists \varepsilon > 0\ \forall \delta > 0 : |x-1| < \delta \Rightarrow e^x < \varepsilon$.

$\diamond$

Answer: (1) false. (2) false. (3) true. (4) false.

Exercise:
(1) Write the definition of the union $A \cup B$ of two sets A and B using the above defined symbols as much as possible.
(2) Write the definition of the intersection $A \cap B$ of two sets A and B using the above defined symbols as much as possible.

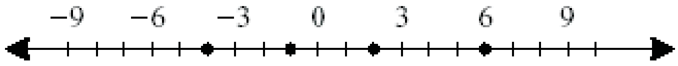
Answer: (1) $x \in A \cup B \Leftrightarrow (x \in A) \vee (x \in B)$. (2) $x \in A \cap B \Leftrightarrow (x \in A) \wedge (x \in B)$.

2. Coordinates in One Dimension

We envision the real numbers as laid out on a line, and we locate real numbers from left to right on this line. If a $<$ b are real numbers then a will lie to the left of b on this line (see fig 2).

Example: On a real number line, plot the numbers $-4, -1, 2, 6$. Also plot the sets $S = \{x \in \mathbb{R} :\ -8 \leq x < -5\} = [-8, -5)$ and $T = \{t \in \mathbb{R} : 7 < t \leq 9\} = (7, 9]$. Label the plots.

Solution: Figure 3 exhibits the indicated points and the two sets. These sets are called half-open intervals because each set includes one endpoint and not the other. $\diamond$



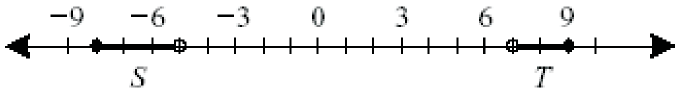


FIG. 3. Numbers and half-intervals on real axis

NOTE 2.1. *The notation* $S = \{x \in \mathbb{R} : \ -8 \le x < -5\} = [-8, -5)$ *is called set builder notation. It says that* S *is the set of all real numbers* x *such that* x *is greater than or equal to* -8 *and less than* -5.

DEFINITION 2.1. *If an interval contains both its endpoints, then it is called a closed interval. If an interval omits both its endpoints, then it is called an open interval. See Fig. 4.*

3. Equations and Inequalities

Properties of equality.

(1) if $a = b$ then $\forall c : a + c = b + c$.
(2) if $a = b$ then $\forall c : a - c = b - c$.
(3) if $a = b$ then $\forall c : a \times c = b \times c$.
(4) if $a = b$ then $\forall c \neq 0 : a \div c = b \div c$.

Example: Find the set of points that satisfy $x - 2 = 4$.
Solution: Using the first property we solve the equality to obtain $x - 2 + 2 = 4 + 2$ or $x = 6$. $\diamond$

Inequality properties.
For a, b, c any real numbers.

(1) if $a < b$ and $b < c$ then $a < c$.
(2) if $a < b$ then $a + c < b + c$.
(3) if $a < b$ then $a - c < b - c$.

Interval Notation

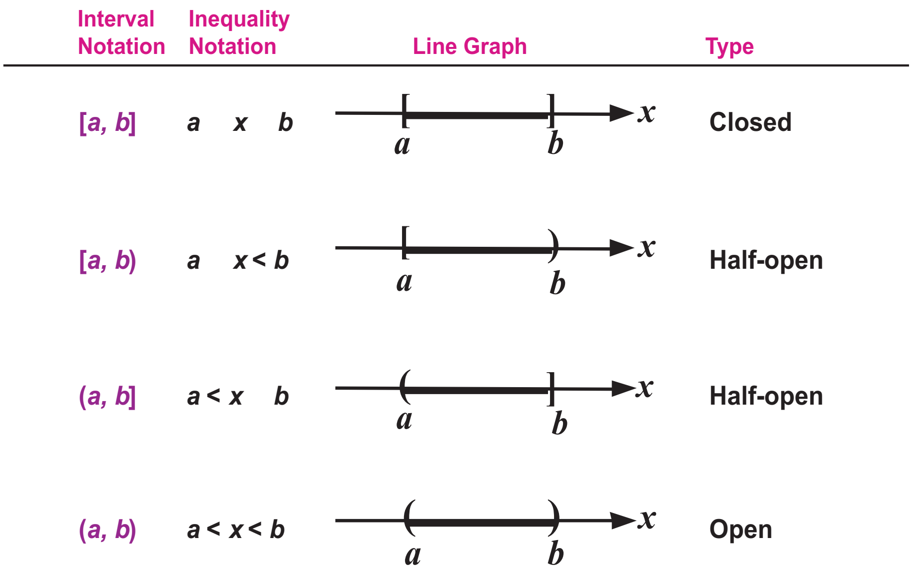


FIG. 4. Interval notation

(4) if $a < b$ and $c > 0$ then $a \times c < b \times c$.
(5) if $a < b$ and $c < 0$ then $a \times c > b \times c$.
(6) if $a < b$ and $c > 0$ then $a \div c < b \div c$.
(7) if $a < b$ and $c < 0$ then $a \div c > b \div c$.

Example: Find the set of points that satisfy $x - 2 < 4$ and exhibit it on a number line.

Solution: We solve the inequality to obtain $x < 6$. The set of points satisfying this inequality is exhibited in Fig. 5. $\diamond$

Exercise: Find the set of points that satisfy the sistem of unequalities

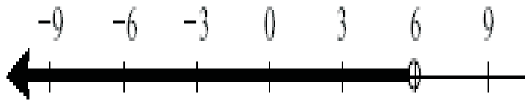


FIG. 5. The solution of unequality $x - 2 < 4$

$$\begin{cases} 2x + 3 > 0; \\ 5 - x \geq 0. \end{cases}$$

Answer:$-\frac{3}{2} < x \leq 5$.

4. Absolute Value. The Distance in the Set of Real Numbers

DEFINITION 4.1. *The absolute value $|x|$ of a number $x \in \mathbb{R}$ is a number defined by setting*

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

DEFINITION 4.2. *The distance between two real numbers x, y is* $|x - y|$.

Properties of absolute value.

(1) $|a| \geq 0$;
(2) $|-a| = a$;
(3) $a^2 = |a|^2$;
(4) $|ab| = |a||b|$;
(5) $-|a| \leq a \leq |a|$;
(6) $|a| = |b| \Leftrightarrow a = \pm b$;
(7) $b > 0, |a| > b \Leftrightarrow a > b$ or $a < b$;
(8) Tiangle unequality: $|a + b| \leq |a| + |b|$.

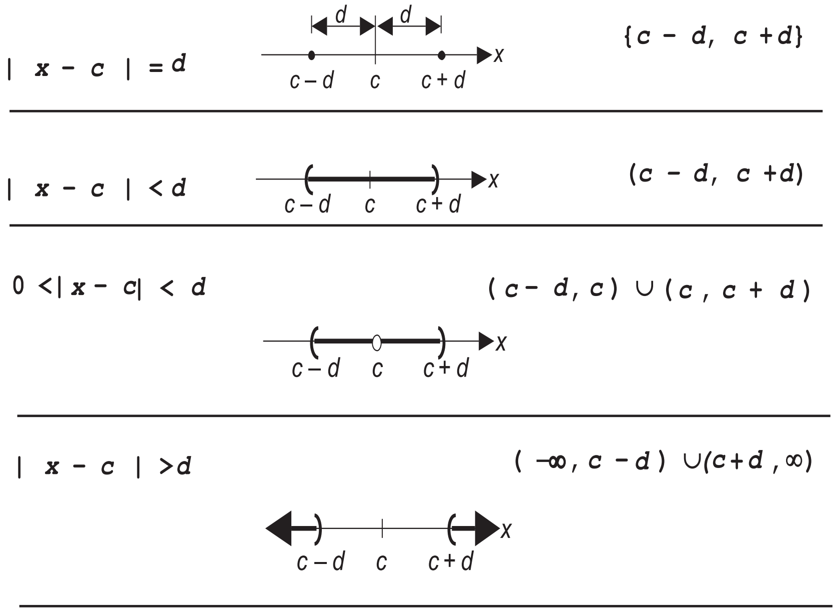


FIG. 6. Absolute value equations and inequalities

Example: Let $x, y, z \in \mathbb{R}$. Show that $|x - y| \leq |x - z| + |z - y|$.
Proof: $|x - y| = |x - z + z - y| \leq |x - z| + |z - y|$ by the triangle inequality. $\diamond$

Example: When do we have inequality in the above estimate? $\diamond$

5. Solving Absolute Value Equation and Inequality

.
See Fig. 6 for scheme of solving the simplest absolute value equation and unequality.

Example: Solve the equation $|2x + 1| = 5$.
Solution: For those value of x for which $2x + 1 \geq 0$ we have
$2x + 1 = 5 \Leftrightarrow 2x = 4 \Leftrightarrow x = 2$.
If $x = 2$ then $2x + 1 = 5 > 0$. Hence $x = 2$ is a solution.

For those value of x for which $2x+1<0$ we have
$|2x+1| = 5 \Leftrightarrow -(2x+1) = 5 \Leftrightarrow -2x = 6 \Leftrightarrow x = -3$.
If $x = -3$ then $2x+1 = -5 < 0$. Hence $x = -3$ is also a solution.
Conclusion: the equation has two solution: $x = 2$ and $x = -3$. $\diamond$

Example: Solve the unequality $|2x+1| \geq 5$.
Solution: By the property 6 of absolute values,
$|2x+1| \geq 5 \Leftrightarrow 2x+1 \geq 5$ or $2x+1 \leq -5$.
$2x+1 \geq 5 \Leftrightarrow 2x \geq 4 \Leftrightarrow x \geq 2$.
$2x+1 \leq -5 \Leftrightarrow 2x \leq -6 \Leftrightarrow x \leq -3$.
Conclusion: solution is $x \geq 2$ or $x \leq -3$. $\diamond$

5.1. Problems

Write without absolute value signs.
5.1. $|\sqrt{3}-2|$.
5.2. $|\sqrt{7}-\sqrt{5}|$.
5.3. $||\sqrt{7}-\sqrt{5}|-|\sqrt{3}-2||$.
5.4. $|x-|1-2x||$ if $x>2$.
5.5. $|\sqrt{3}-\sqrt{|2-\sqrt{15}}|$.
Find all the real solution to equations and inequalities below.

5.6. $|5x-2| = |2x+1|$.
5.7. $|x|+|x-1| = 1$.
5.8. $||x|+1| = 2$.
5.9. $|x-2|+|x-3| = 1$.
5.10. $|x|+|x-1| = 2$.

5.11. $|x+1|+|x+2|-|x-3| = 5$.

5.12. $|x| > 4$.
5.13. $|x+2| \leq 4$.
5.14. $|4x+5| < 3$.
5.15. $|2x-1| \geq 3$.
5.16. $|2x-5| \leq x$.
5.17. $|2x-1| < |3x+1|$.
5.18. $|3x-2| > |2x+1|$.
5.19. $|3+x| \geq |x|$.

5.20. $2|x-3|+|x+1| \leq 3x+1$.

5.2. Answers

5.1. $2-\sqrt{3}$. **5.2.** $\sqrt{7}-\sqrt{5}$. **5.3.** $\sqrt{7}-\sqrt{5}-\sqrt{3}+2$. **5.4.** $x-1$. **5.5.** $\sqrt{3}-\sqrt{\sqrt{15}-2}$. **5.6.** $\{2;3\}$. **5.7.** $\{0;1\}$. **5.8.** $\{-1;1\}$. **5.9.** $2 \le x \le 3$. **5.10.** $\{-\frac{1}{2};\frac{3}{2}\}$. **5.11.** $\{-11;\frac{5}{2}\}$. **5.12.** $x > 4$ or $x < -4$. **5.13.** $-6 \le x \le 2$. **5.14.** $-2 < x < -\frac{1}{2}$. **5.15.** $x \ge 2$ or $x \le -1$. **5.16.** $\frac{5}{3} \le x \le 5$. **5.17.** $x < -2$ or $0 < x < \frac{1}{2}$ or $x > 2$. **5.18.** $x < -1$ or $-\frac{1}{2} < x < \frac{1}{5}$ or $x > \frac{10}{15}$. **5.19.** $x \ge -\frac{3}{2}$. **5.20.** $x \ge 2$.

CHAPTER 3

FUNCTION

1. Coordinates in Two Dimension

DEFINITION 1.1. *The real Cartesian Plane $\mathbb{R}^2$ is the set of all ordered pairs (x, y) of real numbers.*

We represent the elements of $\mathbb{R}^2$ graphically as follows. Intersect perpendicularly two copies of the real number line. These two lines are the **axes**. Their point of intersection which we label $O = (0, 0)$ is called the **origin** (see the Fig. 1).

We determine the coordinates of the given point P determining the x-displacement, or (signed) distance from the y-axis and then determining the y-displacement, or (signed) distance from the x-axis. We refer to this coordinate system as (x, y)-coordinates or Cartesian coordinate system.

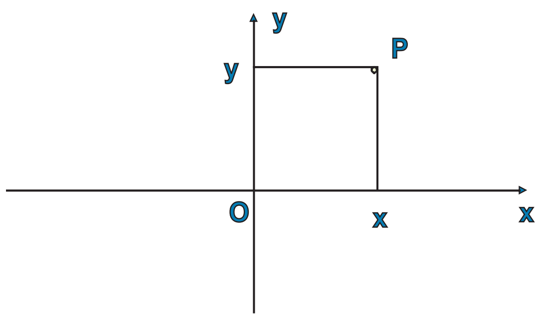


FIG. 1. Cartesian coordinate system

To each point P on the plane we associate an ordered pair $P = (x, y)$ of real numbers. Here x is the **abscissa** (from the Latin *linea abscissa* or line cut-off), which measures the horizontal distance of our point to the origin, and y is the **ordinate**, which measures the vertical distance of our point to the origin. The points x and y are the **coordinates** of P.

This manner of dividing the plane and labelling its points is called the **Cartesian coordinate system**. The horizontal axis is called the **x-axis** and the vertical axis is called the **y-axis**. It is therefore sufficient to have two numbers x and y to completely characterise the position of a point $P = (x, y)$ on the plane $\mathbb{R}^2$.

Example: Plot the points $P = (3, -2), Q = (-4, 6), R = (2, 5), S = (-5, -3)$. Solution: Let's plot the poin P. The first coordinate 3 of the point P tells us that the point is located 3 units to the right of the y-axis. The second coordinate -2 of the point P tells us that the point is located 2 units below the x-axis (because -2 is negative). Plotting of points Q, R, S is similar. See Fig. 2, 3. $\diamond$

Example: Sketch the region $R = \{(x, y) \in \mathbb{R}^2 : \ 1 < x < 3, \ 2 < y < 4\}$.

Solution: The region is a square, excluding its boundary. The graph is shewn in the Figure 4, where we have dashed the boundary lines in order to represent their exclusion.

$\diamond$

Exercise: Sketch the following regions on the plane.
1) $R_1 = \{(x, y) \in \mathbb{R}^2 : \ x < 2\}$.
2) $R_2 = \{(x, y) \in \mathbb{R}^2 : \ |x| \geq 3, |y| \geq 4\}$.
3) $R_3 = \{(x, y) \in \mathbb{R}^2 : \ |x| < 2, y > -4\}$.

2. Definition of Function

DEFINITION 2.1. *Given sets A and B. A* **function** $f : A \to B$ *is a rule which assigns an element $f(a)$ of the set B for every a in A.*

If the sets A and B are finite, then this rule can be expressed in terms of a table or a diagram. Usually the sets A and B are not

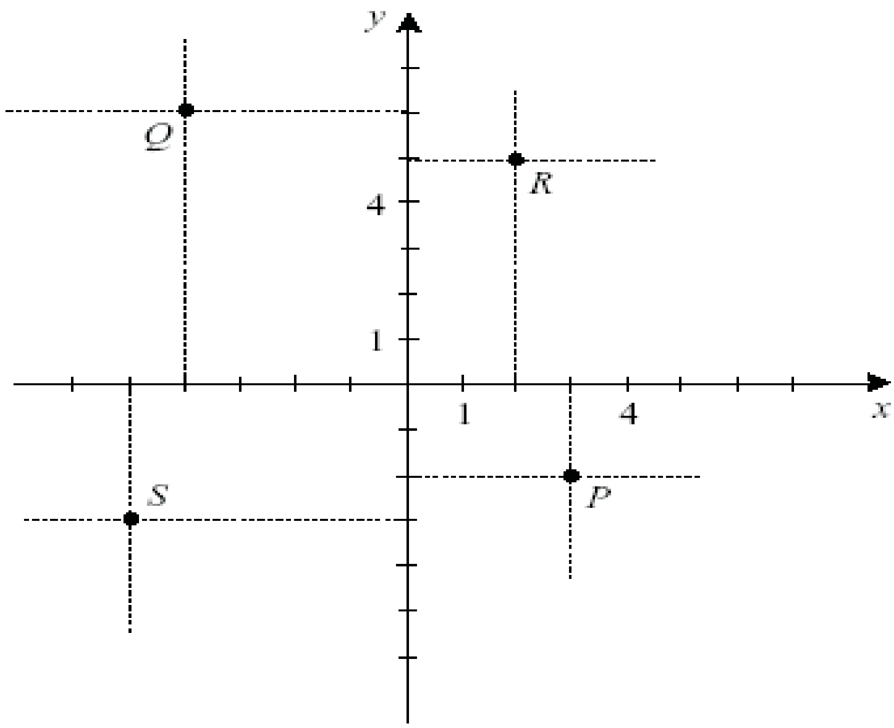


FIG. 2. Points $P = (3, -2), Q = (-4, 6), R = (2, 5), S = (-5, -3)$ on the coordinate plane.

finite. In such a case the rule in question is usually expressed in terms of an algebraic expression, involving possibly special functions, for $f(a)$. Alternatively the rule to compute $f(a)$, for a given a, may be a program taking a as input and producing $f(a)$ as its output.

DEFINITION 2.2. *Let $f : A \to B$ be a function. The set A is the* **domain of definition** *of the function f. The set B is the* **target domain** *of the function f. The set $f(A) = \{f(a)|\ a \in A\} \subset B$ is the* **range** *of the function f.*

Example: Find the natural domain of the rule $y = \frac{1}{x^2 - x - 6}$.

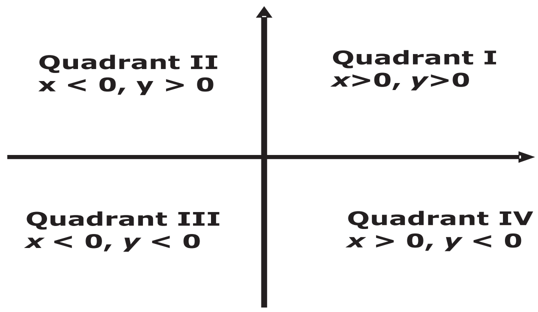


FIG. 3. Quadrants of the plane

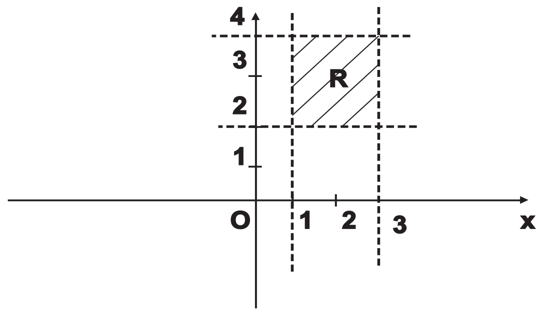


FIG. 4. The region R

Solution: In order for the output to be a real number, the denominator must not vanish. We must have $x^{2\smile}x-6=(x+2)(x-3)\neq 0$, and so $x\neq -2$ nor $x\neq 3$. Thus the natural domain of this rule is $\mathbb{R}\setminus\{-2,3\}$.

Here we used the next rule: if we have an equation $ax^2+bx+x=0$ and the discriminant $D=b^2-4ax\geq 0$ then the solutions x_1, x_2 of the equation are $x_1=\frac{-b+\sqrt{D}}{2a}$, $x_2=\frac{-b-\sqrt{D}}{2a}$ and $ax^2+bx+x=a(x-x_1)(x-x_2)$. $\diamond$

We will usually concern a function $f:\mathbb{R}\to\mathbb{R}$ defined in terms of explicit expression for $f(x)$. The **graph of the function** f is the set $\{(x,f(x))|\ x\in\mathbb{R}\}$.

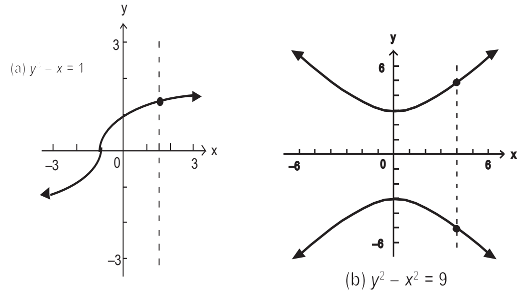


FIG. 5. Vertical line test for a function: a) is a function; b) is not a function

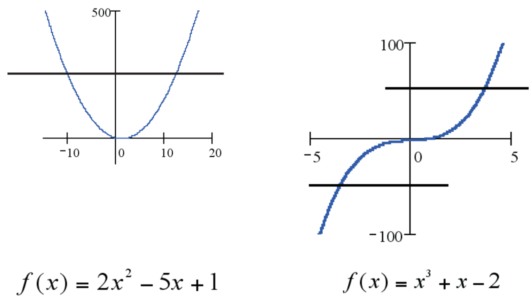


FIG. 6. Horizontal line test for a function: left function is not one-to-one; right function is one-to-one

Vertical line test for a function: An equation defines a function if each vertical line in the rectangular coordinate system passes through at most one point on the graph of the equation. If any vertical line passes through two or more points on the graph of an equation, then the equation does not define a function.

DEFINITION 2.3. *A function $f : A \to B$ is injective or one-to-one if $f(x) = f(y) \Rightarrow x = y$.*

A one-to-one function associates at most one point in the set A to any given point in the set B. **Horizontal Line Test**: If horizontal

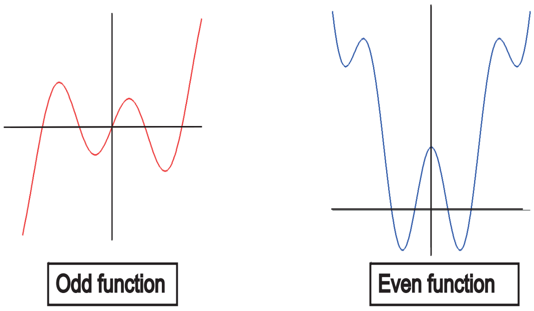


FIG. 7. Odd and even functions

lines intersect the graph of a function f in at most one point, then f is one-to-one.

DEFINITION 2.4. *A function $f : A \to B$ is surjective or onto if $f(A) = B$, i.e. $\forall y \in B \exists x \in A : \; f(x) = y$.*

DEFINITION 2.5. *A function $f : A \to B$ is bijective if it is both one-to-one and surjective. For a bijective function $\forall y \in B \exists ! x \in A : f(x) = y$.*

Observe that the property of being surjective or onto depends on how the set B in the above is defined. Possibly reducing the set B any mapping $f : A \to B$ can always be made surjective.

DEFINITION 2.6. *A function f is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$ for all x.*

The above definition assumes that the domain of definition of the function f is symmetric, i.e. that if the function f is defined at a point x, then it is also defined at the point $-x$. In most of our applications, the functions under consideration are defined for all real numbers.

Example: Investigate is a function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x^3}{x^2+1}$ even, odd, or neither.

Solution: $f(-x) = \frac{(-x)^3}{(-x)^2+1} = -\frac{x^3}{x^2+1} = -a(x)$. Thus the function f is odd. $\diamond$

Exercise: Investigate which of the following functions are even, odd, or neither.

(1) $f(x) = \frac{|x|}{x^2+1}$;
(2) $f(x) = |x| + 2$;
(3) $f(x) = |x + 2|$.

Answer:(1) even; (2) even; (3) neither even nor odd.

THEOREM 2.1. *Let f_1, f_2 be even functions, and let g_1, g_2 be odd functions, all sharing the same common domain.*
Then

(1) *$f_1 \pm f_2$ is an even function.*
(2) *$g_1 \pm g_2$ is an odd function.*
(3) *$f_1 \times f_2$ is an even function.*
(4) *$g_1 \times g_2$ is an even function.*
(5) *$f_1 \times g_2$ is an odd function.*

Proof: We have
1) $(f_1 \pm f_2)(-x) = f_1(-x) \pm f_2(-x) = f_1(x) \pm f_2(x) = (f_1 \pm f_2)(x)$.
2) $(g_1 \pm g_2)(-x) = g_1(-x) \pm g_2(-x) = -g_1(x) \pm g_2(x) = -(g_1 \pm g_2)(x)$.
3) $(f_1 \times f_2)(-x) = f_1(-x) \times f_2(-x) = f_1(x) \times f_2(x) = (f)_1 \times f_2)(x)$.
4) $(g_1 \times g_2)(-x) = g_1(-x) \times g_2(-x) = (-g_1(x)) \times (-g_2(x)) = (g_1 \times g_2)(x)$.
5) $(f_1 \times g_1)(-x) = f_1(-x) \times g_1(-x) = -f_1(x) \times g_1(x) = (f_1 \times g_1)(x)$. $\diamond$

2.1. Symmetry

A graph is said to be **symmetric with respect to the x-axis** if for every point (x, y) on the graph, the point $(x, -y)$ is on the graph.

A graph is said to be **symmetric with respect to the y-axis** if for every point (x, y) on the graph, the point $(-x, y)$ is on the graph. Graph of an even function is symmetric with respect to the y-axis.

o A graph is said to be **symmetric with respect to the origin** if for every point (x, y) on the graph, the point $(-x, -y)$ is on the

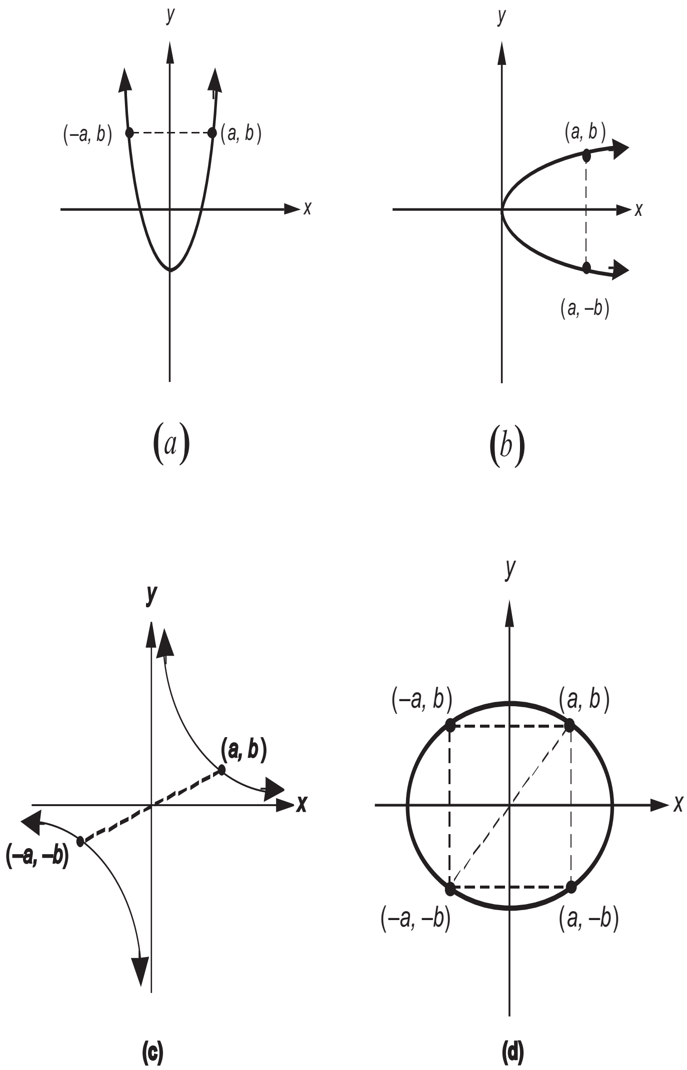


FIG. 8. (a): symmetry with respect to the y-axis; (b): symmetry with respect to the x-axis; (c),(d): symmetry with respect to the origin

graph. Graph of an odd function is symmetric with respect to the origin.

Tests for Symmetry.

x-axis Replace y by $-y$ in the equation. If an equivalent equation results, the graph is symmetric with respect to the x-axis.

y-axis Replace x by $-x$ in the equation. If an equivalent equation results, the graph is symmetric with respect to the y-axis.

origin Replace x by $-x$ and y by $-y$ in the equation. If an equivalent equation results, the graph is symmetric with respect to the origin.

Exercise: Test functions below for symmetry.

(1) $y = \frac{x^2}{4-x^2}$.
(2) $\frac{x^2-1}{x^3}$.
(3) $\frac{|x|}{x^4+1}$.
(4) $|x|x$.
(5) $\sqrt{x} + 2$.

The property of being either odd or even can simplify greatly computations regarding a given function.

A polynomial is even (odd) if all of its terms have even (odd) power. Hence every polynomial is a sum of an even polynomial and an odd polynomial.

THEOREM 2.2. *Every function is a sum of an even function and an odd function assuming that the domain of definition of the function is symmetric.*

Proof: Let f be a function. Define the functions f_+ and f_- be setting $f_+(x) = \frac{f(x)+f(-x)}{2}$ and $f_+(x) = \frac{f(x)-f(-x)}{2}$.

$f_+(-x) = \frac{f(-x)+f(-(-x))}{2} = \frac{f(-x)+f(x)}{2} = f(x)$, i.e. f_+ is even. Similar computation shows that f_- is odd. Finally observe that $f = f_+ + f_-$. $\diamond$

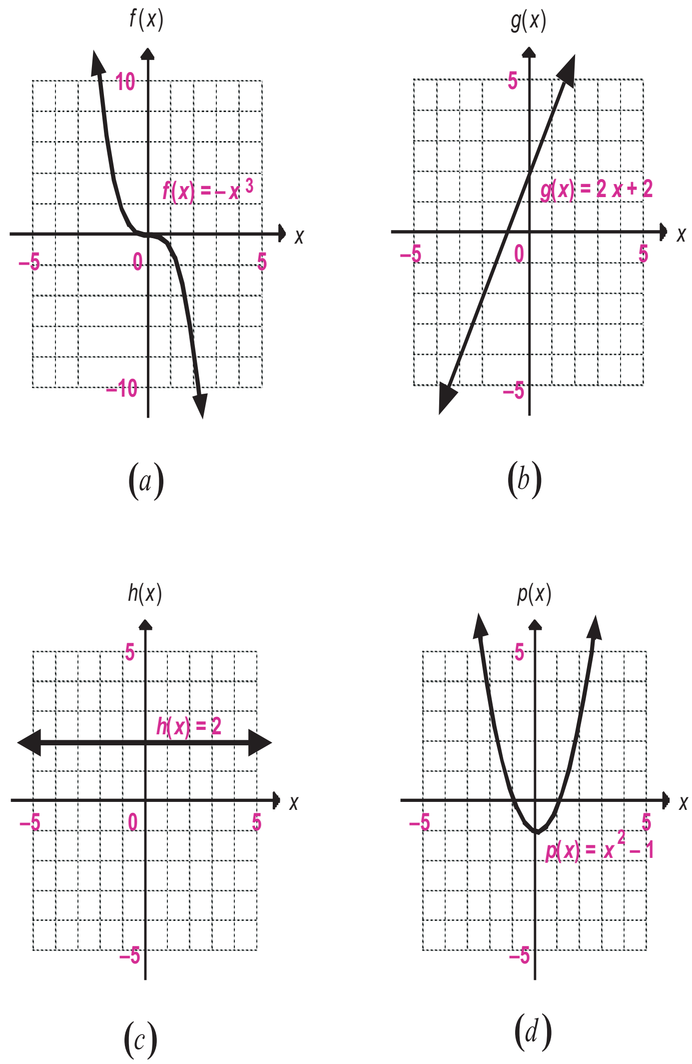


FIG. 9. (a) decreasing function; (b) increasing function; (c) constant function; (d): decreasing for $x < 0$ and increasing for $x > 0$.

2.2. Monotonous Functions

DEFINITION 2.7. *A function f is increasing if $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.*

DEFINITION 2.8. *A function f is decreasing if $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.*

DEFINITION 2.9. *A function f is monotonous if it is either increasing or decreasing.*

In some other texts, functions which are increasing in the above sense, are called strictly increasing. The same applies to the decreasing functions.

Like in the case of even and odd functions, any sufficiently smooth function can be expressed as a sum of an increasing function and a decreasing function. This is a deep fact of analysis.

Observe that monotonous functions are injective or one-to-one but that there are injective functions which are not monotonous.

3. Linear Polynomial Functions

Graphs of linear polynomials $y = ax + b$ are straight lines. The coefficient a determines the angle at which the line intersects the x–axis (**slope**).

Equation $y = b$ defines a horizontal line, equation $x = a$ defines a vertical line.

THEOREM 3.1. *Two lines with slopes m_1 and m_2 are parallel iff $m_1 = m_2$, and perpendicular iff $m_1 m_2 = -1$.*

See Figure 10 for the geometric interpretation of slope.

Example: Mindplotting $y = 2x + 5$. (see Fig. 11).
Step 1: Make a table of friendly values.
Step 2: Plot the points.
Step 3: Draw the straight line throught the points.

$\diamond$

As part of plotting, determine the **intercepts**, those points where the graph crosses the axes:

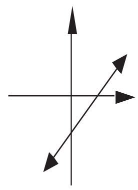
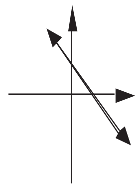
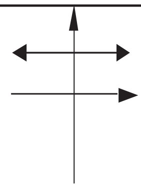
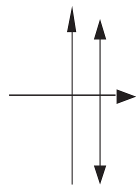
Line	Slope	Graph
Rising as *x* moves from left to right	Positive	
Falling as *x* moves from left to right	Negative	
Horizontal	0	
Vertical	Not defined	

FIG. 10. Geometric interpretation of slope

x	y
0	5
- 2.5	0

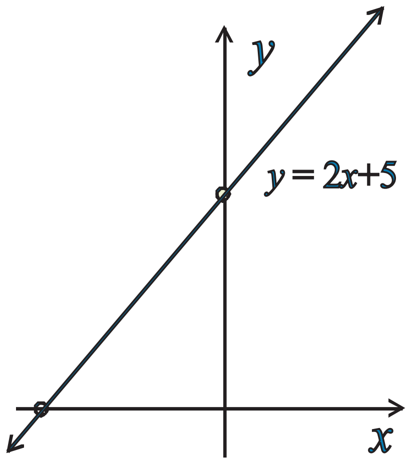


FIG. 11. Graph of function $y = 2x + 5$.

y-intercept: graph crosses the y-axis; *x-intercept*: graph crosses the x-axis.

The table in the picture 11 presents $x-$ and y-intercepts of graph of $y = 2x + 5$.

If straight line is defined by equation $y = mx + b$ then *y-intercept* has coordinates $(0, b)$.

In addition to the **slope − interseptform** $y = mx+b$ of equation of straight line we will use also

Point-Slope Form: the line through the point (x_1, y_1) with slope m has the equation $y - y_1 = m(x - x_1)$.

3.1. Problems

3.1. Find an equation of the line which passes through the point $(-2, 3)$ and has slope 3.

3.2. Find an equation of the line which passes through the point $(1, -2)$ and is parallel to the line $y = 5x - 3$.

3.3. Find an equation of the line which passes through the point $(2, -4)$ and has y-intercept 5.

3.4. Find an equation of the line which passes through the points $(-1, -1)$ and has x-intercept 3.

3.5. What is the slope of the line $x/a + y/b = 1$?

3.6. If the point $(a, -a)$ lies on the line $-2x + 3y = 30$, find the value of a.

3.7. Find the equation of the straight line joining $(3, 1)$ and $(-5, -1)$.

3.8. Find the equation of the straight line parallel to the line $8x - 2y = 6$ and passing through the point $(5, 6)$.

3.9. Find the equation of the line passing through $(12, 0)$ and normal to the line joining $(1, 2)$ and $(-3, -1)$.

3.2. Answers

3.1.$y = 3x + 11$. **3.2.**$y = 5x - 7$. **3.3.**$y = -4.5x + 5$. **3.4.**$y = \frac{1}{4}x - \frac{3}{4}$. **3.5.**$\frac{b}{a}$. **3.6.**$-6$. **3.7.**$y = 3x - 2$. **3.8.**$y = 4x - 14$. **3.9.**$y = \frac{4}{3}x + 16$.

4. Power Functions

Definition 4.1. *A power function of degree n ia a function of the form $f(x) = ax^n$, where $a \in \mathbb{R}$, $a \neq 0$ and $n \in \mathbb{N}$.*

Example:(The identity function) Consider the function $Id : \mathbb{R} \to \mathbb{R}$ such that $Id(x) = x$. This function assigns to every real its own value. The graph of identity function is a straight line, and it is given in figure refidp. $\diamond$

Example: The next base function is the function $y : \mathbb{R} \to \mathbb{R}$ such that $y(x) = x^2$. To plot the graph of the function it is convenient to make a table of values. The table and the graph called parabola are given in the fig. 13. $\diamond$

Example: Let's consider a function $y : \mathbb{R} \to \mathbb{R}$ such that $y(x) = x^3$. The table of values and the graph of the function called cubic are given in the fig. 14. $\diamond$

Propery of power function $f(x) = ax^n$, with n is even.

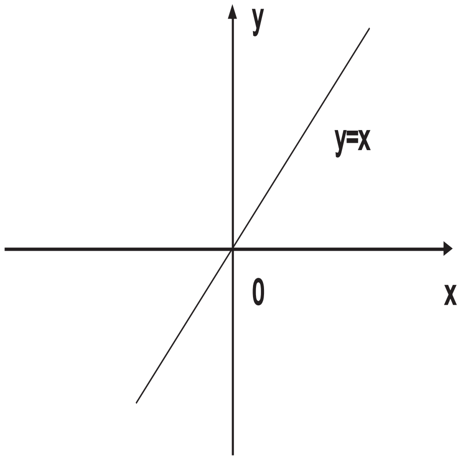


FIG. 12. The identity function

x	$y = x^2$
−3	9
−2	4
−1	1
0	0
1	1
2	4
3	9

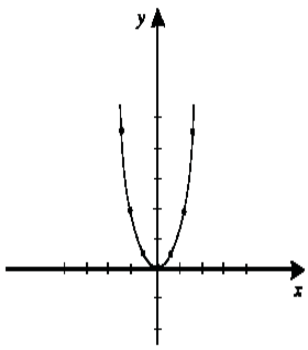


FIG. 13. Graph of the function $y = x^2$ (parabola)

(1) The graph is symmetric with respect to the y-axis, so f is even.
(2) The domain is the set of all real numbers. The range is the set of nonnegative numbers.
(3) The graph always contains the points (-1,1), (0,0), and (1,1).

x	$y = x^3$
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

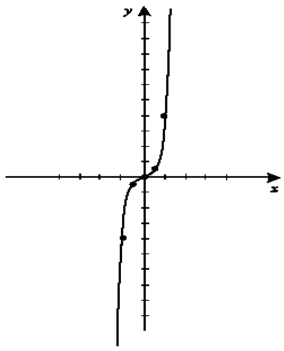


FIG. 14. Graph of the function $y = x^3$ (cubic)

(4) As the exponent increases in magnitude, the graph becomes more vertical when $x < -1$ or $x > 1$, but for x near the origin the graph tends to flatten out and lie closer to the x-axis.

Property of power function $f(x) = ax^n$, with n is odd.

(1) The graph is symmetric with respect to the origin, so f is odd.
(2) The domain and range are the set of all real numbers.
(3) The graph always contains the points $(-1, -1)$, $(0, 0)$, and $(1, 1)$.
(4) As the exponent increases in magnitude, the graph becomes more vertical when $x > 1$ or $x < -1$, but for x near the origin the graph tends to flatten out and lie closer to the x-axis.

5. Piecewise Defined Functions

Sometimes it is necessary to define a function by giving several expressions, for the function, which are valid on certain specified intervals. Such a function is a piecewise defined function.

The absolute value $|x|$ is an example of a piecewise defined function. We have $|x| = x$ if $x \geq 0$ and $|x| = -x$ otherwise. Computations with the absolute value have to be done using its definition as a piecewise defined function.

Example: Express $y = |1-|x-2||$ as a piecewise defined function.

Solution: We have to strip the absolute values from the expression by starting with the innermost absolute values. Observe first that

$$y(x) = \begin{cases} |1-(x-2)| & \text{if } x \geq 2, \\ |1-(2-x)| & \text{if } x < 2, \text{ i.e.} \end{cases}$$

$$y(x) = \begin{cases} |3-x| & \text{if } x \geq 2, \\ |x-1| & \text{if } x < 2. \end{cases}$$

Next observe that

$$|3-x| = \begin{cases} x-3 & \text{if } x \geq 3, \\ 3-x & \text{if } x < 3 \end{cases} \quad \text{and}$$

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1, \\ 1-x & \text{if } x < 1. \end{cases}$$

Combine the above to get

$$y(x) = \begin{cases} x-3 & \text{if } x \geq 3, \\ 3-x & \text{if } 2 \leq x < 3, \\ x-1 & \text{if } 1 \leq x \geq 2, \\ 1-x & \text{if } x < 1. \end{cases}$$

◇

Exercise: Draw the graph of the above function.

5.1. Problems

Draw the graph of functions.

5.1. $y = |2x-1|$.

5.2. $y = x + |x|$.

5.3. $y = x|x|$.

5.4. $y = |x-1| + |x+2|$.

6. Composite Functions

Function composition is the application of one function to the results of another. For instance, the functions $f : X > Y$ and $g : Y > Z$ can be composed by computing the output of g when it has an argument of $f(x)$ instead of x. Intuitively, if z is a function g of y and

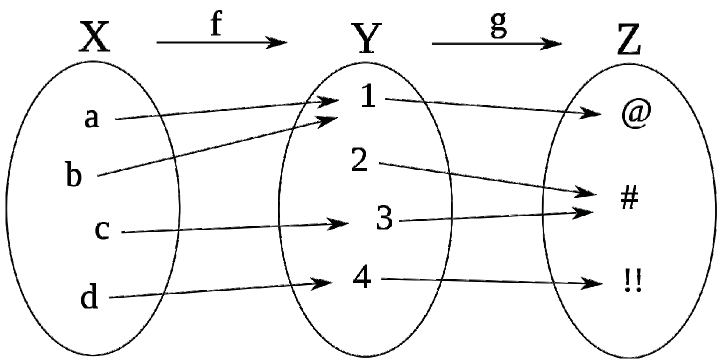


FIG. 15. $g \circ f$, the composition of f and g. For example, $(g \circ f)(c) = \#$.

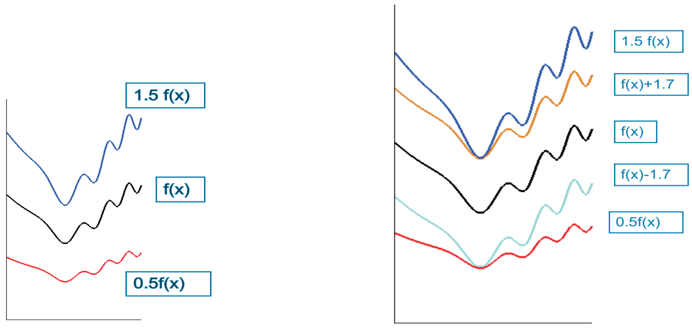
y is a function f of x, then z is a function of x. Thus one obtains a **composite function** $g \circ f : X \to Z$ defined by $(g \circ f)(x) = g(f(x))$ for all $x \in X$ (see fig. 15). The notation $g \circ f$ is read as "g circle f or "g composed with f "g after f "g following f or just "g of f".

Assume that f and g are functions for which the composed function $h = f \circ g$ is defined. Then below properties hold.

- If both f and g are increasing, then also h is increasing.
- If f is increasing and g decreasing, then h is decreasing.
- If f is decreasing and g increasing, then h is decreasing.
- If both f and g are decreasing, then h is increasing.

7. Simple Deformations

Let f be a given function, and let a be a real number. The picture in fig. 16 (a) illustrates how the graph of the function f gets deformed as we replace the values $f(x)$ by $af(x)$. By multiplying the function by a positive constant a the graph gets stretched in the vertical direction if $a > 1$ and squeezed if $0 < a < 1$. By multiplying the function by a negative constant a the graph gets first reflected about the x-axis and then stretched in the vertical direction if $a < -1$ and squeezed if $-1 < a < 0$.



(*a*) (*b*)

FIG. 16. Simple deformations

The effect, on the graph, of multiplying a function with a constant is either stretching, squeezing or, if the constant is negative, then first reflecting and then stretching or squeezing.

Adding a constant to a function means a vertical translation in the graph. The The picture in fig. 16 (b) illustrates this situation.

Let f be a given function, and let $b > 0$ be a real number.

The picture in fig. 17 illustrates how the graph of the function f gets deformed as we replace the values $f(x)$ by $f(x+b)$ and $f(x+b)$.

Example: Using simple deformations of base parabola $y = x^2$, draw the graph of functions $y = x^2 + 2$; $y = x^2 - 3$; $y = -x$; $y = (x+2)^2$; $y = (x-3)^2$; $y = 2x^2$; $y = \frac{1}{2}x^2$.

Solution: The answer is shewn in fig. 18, 19. To get the graph of $y = x^2 + 2$ we should shift the graph of $y = x^2$ up two units. To get the graph of $y = x^2 - 3$ we should shift the graph of $y = x^2$ down tree units. To get the graph of $y = (x+2)^2$ we should shift the graph of $y = x^2$ left two units. To get the graph of $y = (x-3)^2$ we should shift the graph of $y = x^2$ right three units. $\diamond$

Example: Show each stage to obtain the graph of function $y(x) = 4 - 2(x-1)^4$.

Solution is shewn in the fig. 20. $\diamond$

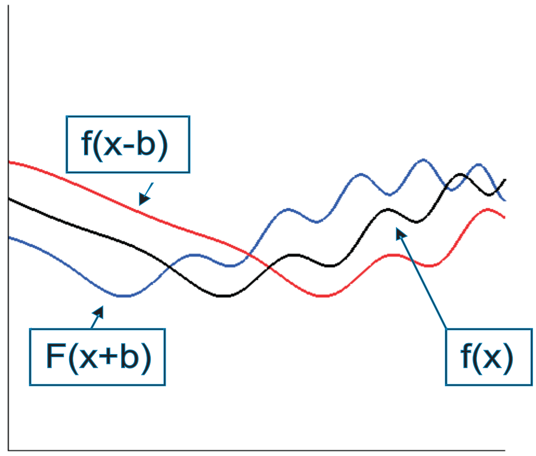


FIG. 17. Graphs of functions $f(x)$, $f(x+b)$ and $f(x+b)$

Example: Use simple deformation th draw the graph of function:

$y = (x-3)^2 + 4;$ $y = -|x+1| + 5.$

◇

8. Quadratic Function

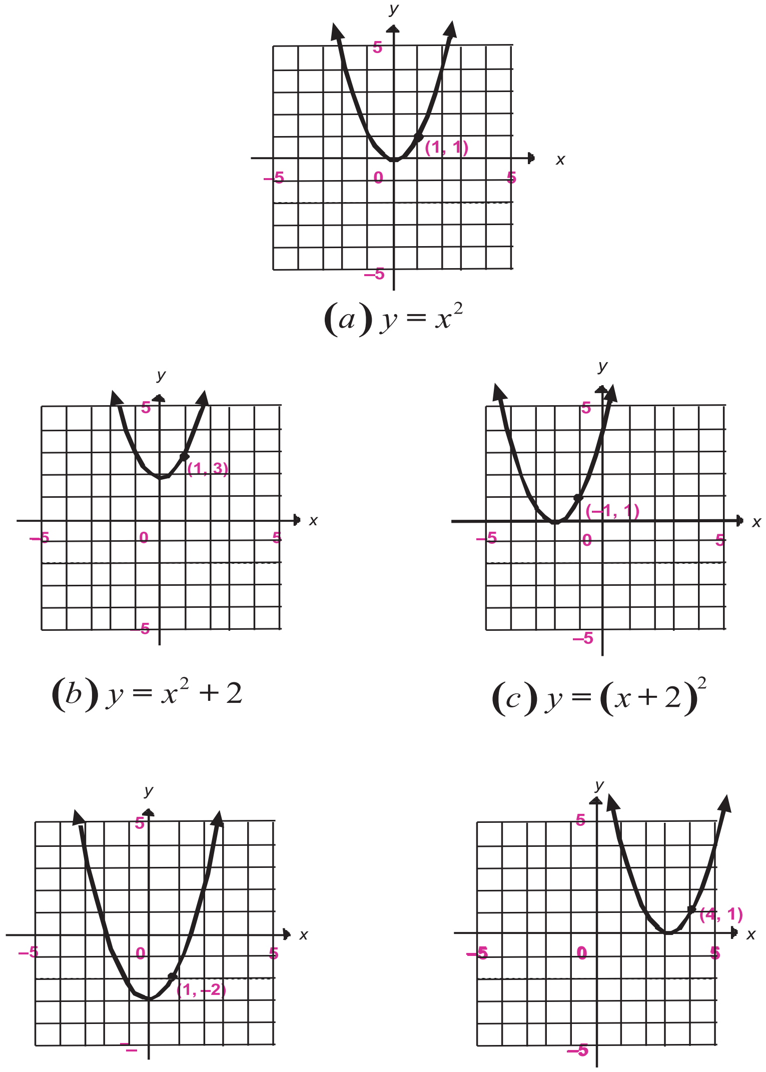
8.1. Quadratic Equation

A **quadratic equation** is an equation equivalent to one of the form $ax^2 + bx + c = 0$ where $a \neq 0$, b, c are real number.

Example: Here are some examples of quadratic equation: $x^2 - 9 = 0$, $x^2 + 2x + 1 = 0$, $x^2 - x + 1 = 0$, $-x^2 + 2x = 0$. ◇

Methods for Solving Quadratic Equations.

- Factoring.
- Graphing.
- Square Root Method.
- Complete the Square Method.
- Quadratic Formula.



(a) $y = x^2$

(b) $y = x^2 + 2$ (c) $y = (x+2)^2$

(d) $y = x^2 - 2$ (e) $y = (x-3)^2$

FIG. 18. Vertical and horizontal shifts

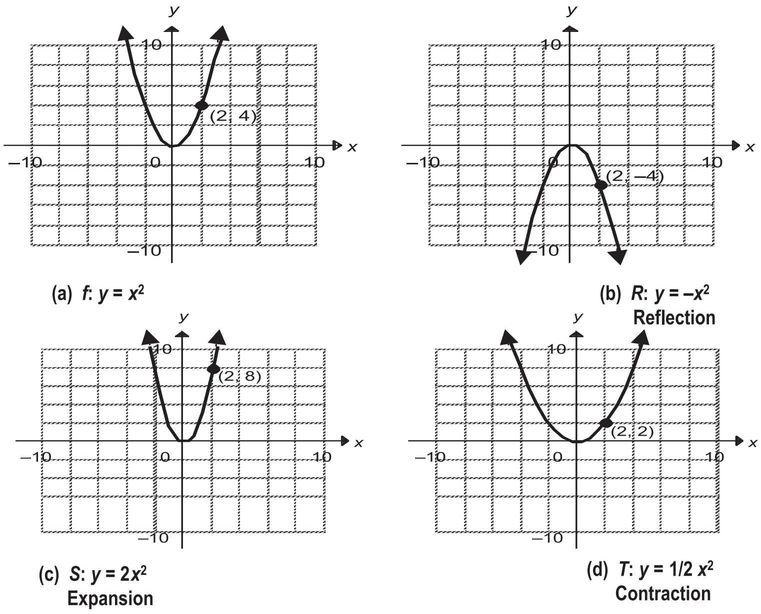


FIG. 19. Reflection, expansion and contraction

The Square Root Method

If $x^2 = p$ and $p \geq 0$, then
$$x = \sqrt{p} \text{ or } x = -\sqrt{p}$$

Example: Solve the equation $(x+1)^2 = 5$.
Solution: using the square root method
$$x + 1 = \sqrt{5} \text{ or } x + 1 = -\sqrt{5}$$
$$x = -1 + \sqrt{5} \text{ or } x = -1 - \sqrt{5}$$
Answer: $x = -1 \pm \sqrt{5}$. ◇

Quadratic Formula

Consider a quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$. The number $D = b^2 - 4ac$ is called a **discriminant**.

- If $D > 0$, there are 2 unequal real solutions.
- If $D = 0$, there is repeated real solution.
- If $D < 0$, there are no real solutions.

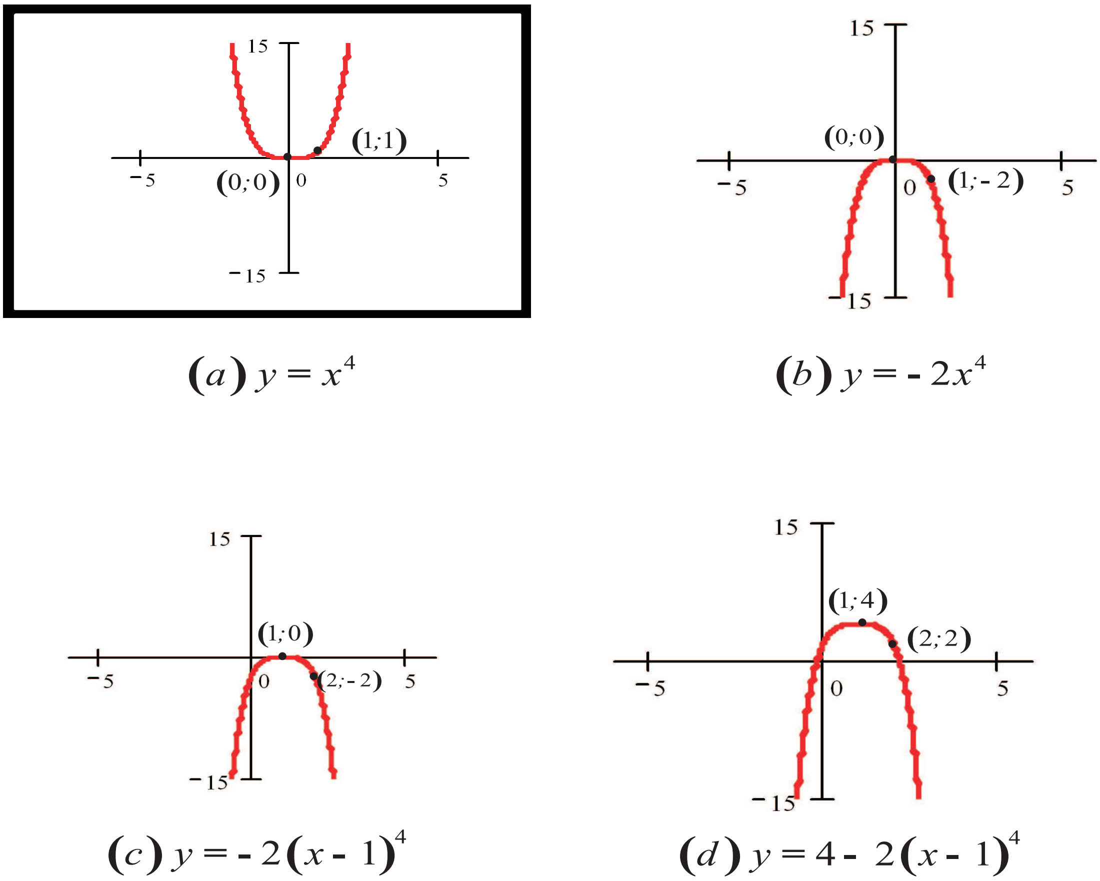


FIG. 20. Stage of drawing of graph of $y = 4 - 2(x-1)^2$

If $D \geq 0$, the solutions of the equation are given by formula

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a} \textit{ and}$$

$$ax^2 + bx + c = a(x - x_1)(x - x_2).$$

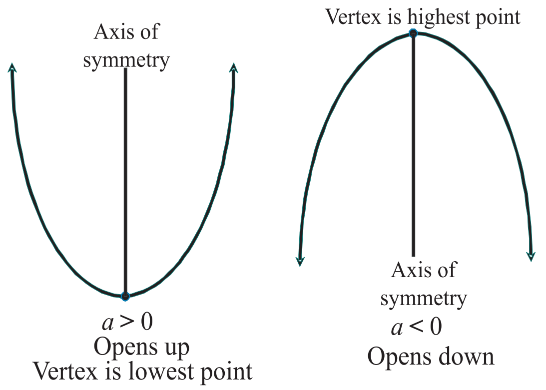


FIG. 21. Graph of the quadratic function
$$y = ax^2 + bx + c$$

8.2. Quadratic Function

A **quadratic function** is a 2nd-degree polynomial function of the form $y = ax^2 + bx + c$ where $a \neq 0$, b, c are real number.

The graph of a quadratic function is a parabola, which **opens up** if $a > 0$ and **opens down** if $a < 0$ (see fig. 21.

The **vertex** of parabola is the lowest for $a > 0$ or highest for $a < 0$ point.

The x-coordinate of the vertex of the parabola is given by $-\frac{b}{2a}$, and the y-coordinate can be found by substituting this value for x into the equation $y = ax^2 + bx + c$.

If the vertex of the parabola has coordinates (h, k), then the equation of the parabola can be reduced to the form $y = a(x - h)^2 + k$.

Parabola has a symmetry axis at the line $x = h$.

The x-intercepts of the parabola, if there are any, are the solutions of the quadratic equation $ax^2 + bx + c = 0$.

Example: Without graphing, locate the vertex and find the axis of symmetry of the parabola $f(x) = -3x^2 + 12x + 1$. Does it open up or down?

Solution: The vertex locate at point (h, k) where

$h = -\frac{b}{2a} = -\frac{12}{2\times(-3)} = 2$,
$k = f(2) = -3 \times 2^2 + 12 \times 2 + 1 = 13$.
The axis of symmetry is $x = 2$.
Since $a = -3 < 0$, the parabola is open down. $\diamond$

Example: Find the vertex of the parabola $f(x) = -3x^2+12x+1$ by completing the square.

Solution: $f(x) = -3x^2 + 12x + 1 = -3(x^2 - 4x) + 1 = -3(x^2 - 4x + 4 - 4) + 1 = -3\left((x-2)^2 - 4\right) + 1 = -3(x-2)^2 + 13$. $\diamond$

Exercise: Find the minimal value of the function $f(x) = 2x^2 + 2x + 5$.

8.3. Steps for Graphing a Quadratic Function by Hand

(1) Determine the vertex.
(2) Determine the axis of symmetry.
(3) Determine the y-intercept, $f(0)$.
(4) Determine the x-intercepts (if there are any).
(5) If there are no x-intercepts determine another point from the y-intercept using the axis of symmetry.
(6) Graph.

Exercise: Determine whether the graph opens up or down. Find its' vertex, axis of symmetry, y-intercept, x-intercept. Draw the graph.

(1) $f(x) = 2x^2 + 12x - 5$;
(2) $f(x) = 2x^2 - 8x + 7$;
(3) $f(x) = -x^2 - 5x + 9$.

8.4. Sign of a Quadratic Function with Application to Inequalities

Example: Solve the inequality $x^2 - 2x - 3 > 0$.

Solution: First, we need to look at the associated function $y = x^2 - 2x - 3$ and consider where its graph is above the x-axis (that is $y > 0$). To do this, we need to know where the graph crosses the

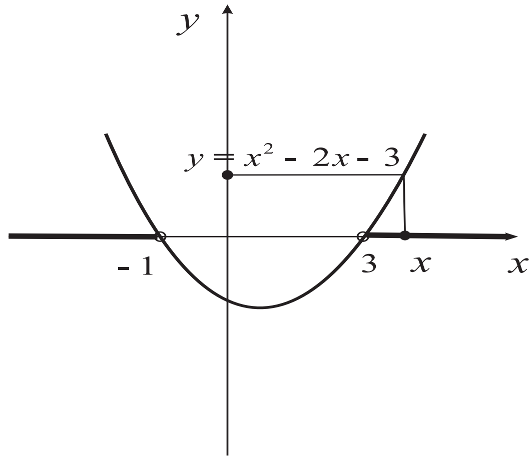


FIG. 22. Graph of function $y = x^2 - 2x - 3$ and solution of inequality $x^2 - 2x - 3 > 0$

x-axis. That is, we first need to find where $y = x^2 - 2x - 3$ is equal to zero:

$$x^2 - 2x - 3 = 0$$
$$(x + 1)(x\breve{}3) = 0$$
$$x = \breve{}1 \text{ or } x = 3$$

This zeroes divide the $x-$axis into three intervals: $(-\infty, -1)$, $(-1, 3)$, $(3, +\infty)$.

Now we need to figure out where (that is, on which intervals) the graph is above the x-axis. Since the parabola opens up, the graph is below the x-axis in the middle, and above the x-axis on the ends: $y < 0$ if and only if $x \in (-\infty, -1)$ or $x \in (3, +\infty)$. So, the solution of the inequality is $x \in (-\infty, -1)$ or $x \in (3, +\infty)$. We can rewrite last conditions as $x < -1$ or $x > 3$.

Answer: $x < -1$ or $x > 3$. $\diamond$

We can apply the concept used in the solution of inequality above to solve of arbitrary quadratic inequality. In general there are three following possibilities (see also fig. 23).

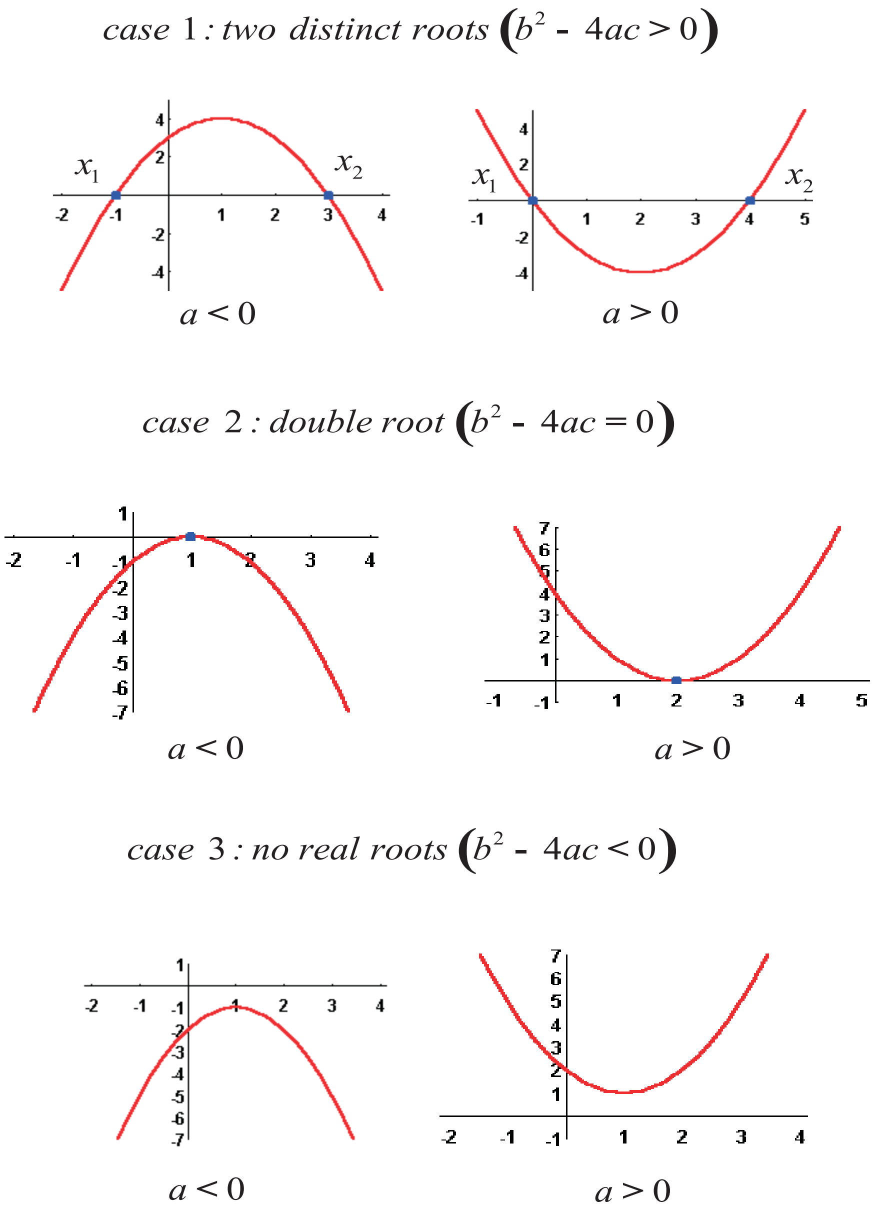


FIG. 23. Sign of a Quadratic Function

1. The equation ax^2+bx+c has two distinct real roots x_1, x_2. Let $x_1 < x_2$. In this case, we have $ax^2+bx+c = a(x-x_1)(x-x_2)$. Since $(x-x_1)(x-x_2)$ is always positive when $x < x_1$ and $x > x_2$ and always negative when $x_1 < x < x_2$, we get ax^2+bx+c has the same sign as the coefficient a when $x < x_1$ and $x > x_2$ and the opposite sign as the coefficient a when $x_1 < x < x_2$.

2. The equation ax^2+bx+c has double root. In this case $ax^2+bx+c = a(x+\frac{b}{2a})^2$, and the expression ax^2+bx+c equal to zero if $x = -\frac{b}{2a}$ and has the same sign as the coefficient a for any $x \neq -\frac{b}{2a}$.

3. The equation ax^2+bx+c has no real roots. In this case the expression ax^2+bx+c has the same sign as the coefficient a for any x.

8.5. Problems

Solve inequalities.

8.1. $x^2 > 0$.
8.2. $x^2 \leq 4$.
8.3. $x^2 > -4$.
8.4. $x^2 - 2x \geq 0$.
8.5. $x^2 > -2x$.
8.6. $x^2 - 6x - 7 < 0$.
8.7. $x^2 - 3x + 5 \geq 0$.
8.8. $-5x^2 + 10x - 5 < 0$.
8.9. $x^2 + 4x + 4 \geq 0$.
8.10. $-2x^2 - x + 1 < 0$.
8.11. $-3x^2 + 2x + 1 < 0$.

8.6. Answers

8.1.$x \neq 0$. **8.2.**$-2 \leq x \leq 2$. **8.3.** any real number. **8.4.**$x \leq 0$ or $x \geq 2$. **8.5.**$x < -2$ or $x > 0$. **8.6.**$-1 < x < 7$. **8.7.** no solution. **8.8.** any real number. **8.9.**$x < -1$ or $x > \frac{1}{2}$. **8.10.**$-\frac{1}{3} < x < 1$.

9. Polynomial Function

DEFINITION 9.1. *A polynomial function (or simply polynomial) is a function of the form* $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ *where* $a_0, a_1, \ldots a_n$ *are real numbers,* $a_0 \neq 0$*, and* n *is a natural number. Number* n *is called the degree of polynomial function.*

Domain of the polynomial function is all real numbers.

Exercise: Determine which of the following are polynomials. For those that are, state the degree.

(1) $f(x) = 3x^2 - 4x + 5$;
(2) $f(x) = \sqrt[3]{x} - 5$;
(3) $f(x) = \frac{3x^5}{5-2x}$.

Answer: (1) polynomial of degree 3; (2), (3) not a polynomial.

A number x for which $P(x) = 0$ is called a **root of the polynomial** P.

THEOREM 9.1. *A polynomial of degree* n *has at most* n *real roots. Polynomials may have no real roots, but a polynomial of an odd degree has always at least one real root.*

9.1. Polynomial Equation. Division Algorithm for Polynomials

The main tools for finging roots of polinomial of degree greater then 2 is a Rational Zeros Theorem, Factor Theorem and polynomial division described below.

THEOREM 9.2 (Rational Zeros Theorem). *Let* $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ *is a polynomial function of power 1 or greater such that every coefficient* a_i *is an integer. If* p/q *in the lowest terms, is a rational zero of* f*, then* p *must be a factor of* a_0 *and* q *must be a factor of* a_n.

Example: Find the roots of equation $x^3 - 2x^2 - x + 2$.

Solution: The equation has at most three roots.

Factors of 2 are $\pm1, \pm2$; factors of 1 are ±1, then the possible root of the equation is among the numbers $\pm1, \pm2$.

By direct sunstitution one can check that $-1, 1$ and 2 are roots of the equation. $\diamond$

THEOREM 9.3 (Factor Theorem). *Let* $P(x)$ *be a polynomial. Then* $P(c) = 0$ *if and only if* $(x - c)$ *is a factor of* $P(x)$.

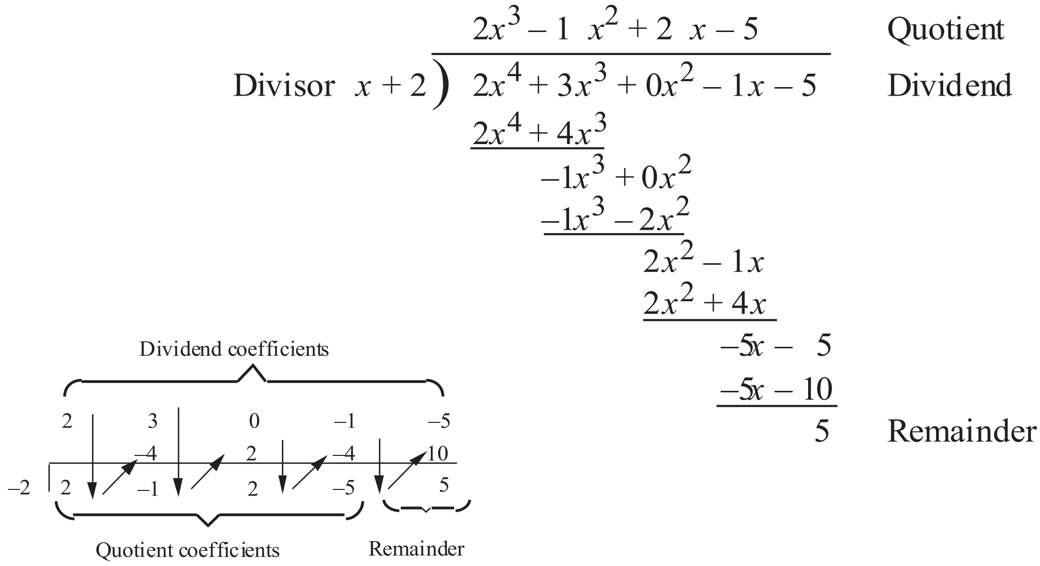


FIG. 24. *PolynomialDivision*

THEOREM 9.4. *If $f(x)$ and $g(x)$ denote polynomial functions and if $g(x)$ is not the zero polynomial, then there are unique polynomial functions $q(x)$ and $r(x)$ such that*

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}, \text{ or}$$

$$f(x) = q(x)g(x) + r(x),$$

where $r(x)$ is either the zero polynomial or a polynomial of degree less than that of $g(x)$.

The algorithm of obtaining functions $q(x)$ and $r(x)$ is shewn in the fig. 24

Example: Solve $x^5 + x^4 - 9x^3 - x^2 + 20x - 12 = 0$.
Solution: The potentional roots are among the numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$. By direct substitution one can get that 1

is a root. Then by factor theorem $x-1$ is a divisor of $P(x) = x^5+x^4-9x^3-x^2+20x-12$. Divide $P(x)$ by $x+3$ to obtain $P(x) = x^5+x^4-9x^3-x^2+20x-12 = (x-1)(x^5+x^4-9x^3-x^2+20x-12)$. Now let's try to find roots of $Q(x) = x^5+x^4-9x^3-x^2+20x-12$. The potentional roots are among the numbers $\pm1, \pm2, \pm3, \pm4, \pm6, \pm12$. By direct substitution one can get that 2 is a root. Divide $Q(x)$ by $x-2$ to obtain $Q(x) = x^5+x^4-9x^3-x^2+20x-12 = (x-2)(x^3+4x^2+x-6)$ and $P(x) = (x-1)Q(x) = (x-1)(x-2)(x^3+4x^2+x-6)$. Similarly, 1 is a root of x^3+4x^2+x-6 and one can get $P(x) = (x-1)(x-2)(x-1)(x^2+5x+6)$ and, finally, $P(x) = (x-1)(x-2)(x-1)(x+2)(x+3) = (x-1)^2(x-2)(x+2)(x+3)$.

Answer: the roots are $1, 2, -2, -3$. $\diamond$

9.2. Graph of Polynomial

The properties of graph of polynomial function are listed in following propositions.

DEFINITION 9.2. *If $(x-r)^m$ is a factor of polynomial $P(x)$ and $(x-r)^{m+1}$ is not a factor of $P(x)$ then r is called a zero of multiplicity r of $P(x)$.*

Example: Let $P(x) = (x-1)^2(x-2)(x+3)^5$. Then 1 is a zero of multiplicity 2, 2 is a zero of multiplicity 1, and -3 is a zero of multiplicity 5. $\diamond$

PROPOSITION 9.1. *If r is a zero of even multiplicity, then sign of $P(x)$ does not change from one side to the other side of r, and the graph* **touches** *x-axis at r.*

PROPOSITION 9.2. *If r is a zero of odd multiplicity, then sign of $P(x)$ changes from one side to the other side of r, and the graph* **crosses** *x-axis at r.*

PROPOSITION 9.3. *If $P(x)$ is a polynomial function of degree n, then $P(x)$ has at most $n-1$ turning points.*

NOTE 9.1. *For large positive and for small negative values of x the graph of the polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ resembles the graph of the power function $f(x) = a_0x^n$.*

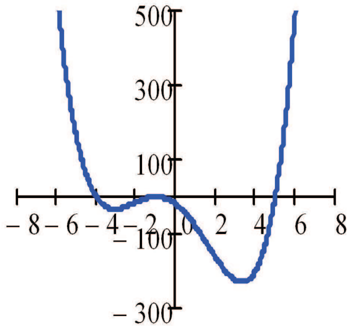


FIG. 25. The graph of polynomial
$$P(x) = (x+1)^2(x-5)(x+4)$$

Steps for graphing a polynomial.

(1) Find the x- and y-intercepts.
(2) Determine whether the graph crosses or touches the x-axis at each x-intercept.
(3) Determine the maximum number of turning points on the graph of P.
(4) Use the x-intercepts and test numbers to find the intervals on which the graph is above the x-axis and the intervals on which the graph is below the x-axis.
(5) Find the power function that the graph of P resembles for large values of x.
(6) Put all the information together, and connect the points with a smooth, continuous curve to obtain the graph.

Example: Draw the graph of $P(x) = (x+1)^2(x-5)(x+4)$.

(1) x-intercepts are $x = -1, x = 5, x = -4$;

y-intercept is $y = P(0) = (0+1)^2(0-5)(0+4) = -20$.

(2) at $x = -1$ the graph touches, at $x = 5, x = -4$ the graph crosses the x-axis.

(3) the maximum number of turning points is 3.

(4) zeroes $x = -1, x = 5, x = -4$ cut the x-axis into four regions:

($-\infty$, -4) the test point $x = -100$: $P(-100) > 0 \Rightarrow$ the graph is above the x-axis for any x from this interval;

(-4,-1) the test point $x = -3$: $P(-3) < 0$;

(-1,5) the test point $x = 0$: $P(0) < 0$;

(5,$+\infty$) the test point $x = 100$: $P(100) > 0$.

(5) for large values of x the graph resembles the graph of $y = x^4$.

(6) See the graph at the fig. 25.

$\diamond$

9.3. Key Steps in Solving Polynomial Inequalities

Step 1. Write the polynomial inequality in standard form (a form where the right-hand side is 0.)

Step 2. Find all real zeros of the polynomial (the left side of the standard form.)

Step 3. Plot the real zeros on a number line, dividing the number line into intervals.

Step 4. Choose a test number (that is easy to compute with) in each interval, and evaluate the polynomial for each number (a small table is useful.)

Step 5. Use the results of step 4 to construct a sign chart, showing the sign of the polynomial in each interval.

Step 6. From the sign chart, write down the solution of the original polynomial inequality (and draw the graph, if required.)

Example: Solve $x^3 - x^2 - 6x > 0$.

Solution: Factoring gives

$$x(x^2 - x - 6) > 0 \textit{ or}$$

$$x(x-3)(x+2) > 0$$

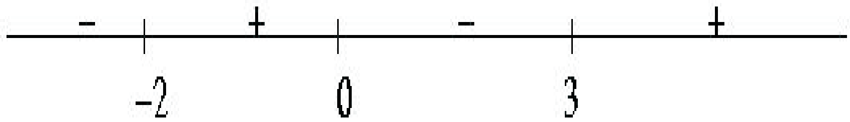


FIG. 26. Solution of $x^3 - x^2 - 6x > 0$

. Mark 0,3, and -2 on a number line and use the test point $x = 1$ for which $x(x-3)(x+2) = -6 < 0$. Since the multiplicity of every zero is odd, we get the sign chart shown in the fig. 26.

Therefore the solution is given by $-2 < x < 0$ or $x > 3$.

$\diamond$

9.4. Problems

Solve inequalities.
9.1. $x^3 - x^2 > 0$.
9.2. $(x-1)(x^2-4) \leq 0$.
9.3. $(x^2-9)(x+1)(x^2+x+1) > 0$.
9.4. $(x-5)(x+4)(x^2+6x+9) \geq 0$.
9.5. $x^3 - 5x^2 - 22x + 56 \geq 0$.

9.5. Answers

9.1.$x > 1$. **9.2.**$x \leq -2$ or $1 \leq x \leq 2$. **9.3.**$3 < x < -1$ or $x > 3$. **9.4.**$x \leq -4$, $x = -3$ or $x \geq 5$. **9.5.**$-4 \leq x \leq 2$ or $x \geq 7$.

10. Rational Inequalities

DEFINITION 10.1. *A rational function is a finction of form* $f(x) = \frac{P(x)}{Q(x)}$ *where* $P(x), Q(x)$ *are polynomials.*

Step 1. Write the inequality in one of the standard form
$f(x) > 0, f(x) < 0, f(x) \geq 0, f(x) \leq 0,$
where $f(x)$ is written as a single quotient.

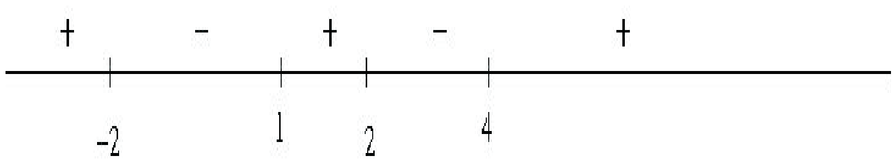


FIG. 27. Solution of $x^3 - x^2 - 6x > 0$

Step 2. Determine the numbers at which the function f equals zero and also those numbers at which it is undefined.

Step 3. Use these numbers to separate the real line into intervals.

Step 4. Choose a test number (that is easy to compute with) in each interval, and evaluate the function f for each number (a small table is useful.)

Step 5. Use the results of step 4 to construct a sign chart, showing the sign of the function f in each interval.

Step 6. From the sign chart, write down the solution of the original inequality. If the inequality is not strict, include the solutions of $f(x) = 0$ in the solution set, but do not include those where f is undefined.

Example: Solve $\frac{x^2-5x-4}{x^2-4} \leq 0$.
Solution: Factoring gives

$$\frac{(x-4)(x-1)}{(x-2)(x+2)} \leq 0.$$

Mark -2, 1,2 and 4 on a number line and use the test point $x = 0$ for which $\frac{(x-4)(x-1)}{(x-2)(x+2)} = -1 < 0$. Since the multiplicity of every zero of numerator and denominator is odd, we get the sign chart shown in the fig. 27.

Since the inequality is not strict, we can include the zeros of the numerator; so the solution is given by $-2 < x \leq 1$ or $2 < x \leq 4$.

$\diamond$

10.1. Problems

Solve the inequalities.

10.1. $\frac{x^4+x^2+1}{x^2-4x-5} < 0.$

10.2. $\frac{x^3-x^2+x-1}{x+8} \leq 0.$

10.3. $\frac{x^2-5x+7}{-2x^2+3x+2} \geq 0.$

10.4. $\frac{x^6+3x^4-x^2-3}{x^3-64x} < 0.$

10.5. $\frac{1}{2-x} + \frac{5}{2+x} < 1.$

10.6. $\frac{x}{x-5} > \frac{1}{2}.$

10.7. $\frac{5x+8}{4-x} < 2.$

10.8. $\frac{1}{x+2} < \frac{3}{x-3}.$

10.2. Answers

10.1. $-1 < x < 5$. **10.2.** $-8 < x \leq 1$. **10.3.** $-\frac{1}{2} < x < 2$. **10.4.** $x < -8$ or $-1 < x < 0$ or $1 < x < 8$. **10.5.** $-2 < x < 2$. **10.6.** $x < -5$ or $x > 5$. **10.7.** $0 < x < 4$. **10.8.** $-\frac{9}{2} < x < -2$ or $x > 3$.

11. Exponential Function

DEFINITION 11.1. *An* **exponential function** *is a function of the form*

$$f(x) = a^x$$

where a is a positive real number $(a > 0)$ and $a \neq 1$. The domain of f is the set of all real numbers.

Summary of the characteristics of the graph of
$$f(x) = a^x, \ a > 1$$

- The domain is all real numbers. Range is set of positive numbers.
- No x-intercepts; y-intercept is 1.
- The x-axis (y=0) is a horizontal asymptote as $x \to -\infty$.
- $f(x) = a^x$, a>1, is an increasing function and is one-to-one.
- The graph contains the points (0,1); (1,a), and (-1, 1/a).
- The graph is smooth continuous with no corners or gaps.

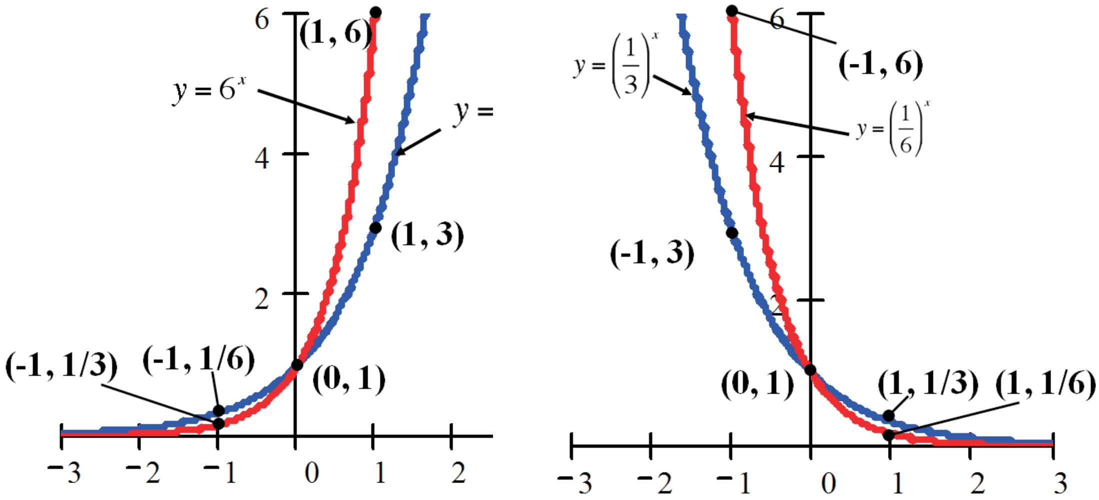


FIG. 28. Graph of the exponential function

Summary of the characteristics of the graph of
$f(x) = a^x,\ 0 < a < 1$

- The domain is all real numbers. Range is set of positive numbers.
- No x-intercepts; y-intercept is 1.
- The x-axis (y=0) is a horizontal asymptote as $x \to +\infty$.
- $f(x) = a^x,\ 0 < a < 1$, is a decreasing function and is one-to-one.
- The graph contains the points (0,1); (1,a), and (-1, 1/a).
- The graph is smooth continuous with no corners or gaps.

Exercise: Graph $f(x) = 3^{-x}+2$ and determine the domain, range and horizontal asymptote of f.

Answer:
Graph: see figure 29
Domain: all real numbers
Range: $\{y|y > 2\}$ or $(2, +\infty)$
Horizontal Asymptote: y = 2.

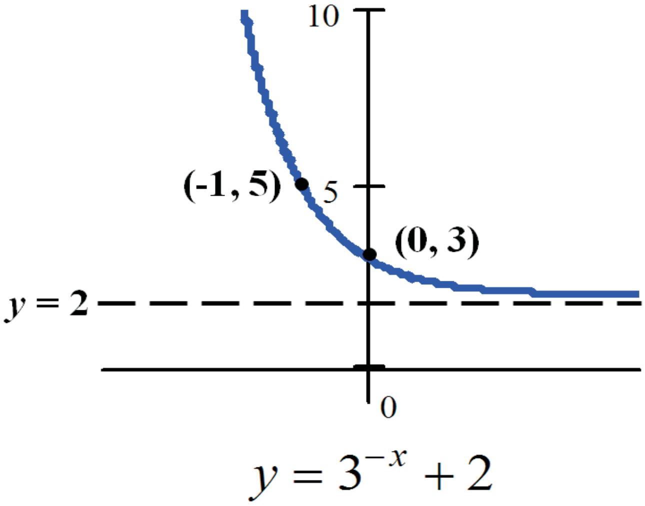


FIG. 29. Graph of the function $f(x) = 3^{-x} + 2$

11.1. The Number e

From the picture it appears obvious that, as the parameter a grows, also the slope of the tangent, at $x = 0$, of the graph of the function a^x grows.

DEFINITION 11.2. *The mathematical constant* **e** *is defined as the unique number e for which the slope of the tangent*[1] *of the graph of e^x at $x = 0$ is 1.*

$$e \approx 2.718281828.$$

The number e is defined also as the number that the expression

$$\left(1 + \frac{1}{n}\right)^n \textit{ approaches as } n \to \infty.$$

[1]The slope of a tangent line is the tangent of the angle at which the tangent line intersects the x-axis.

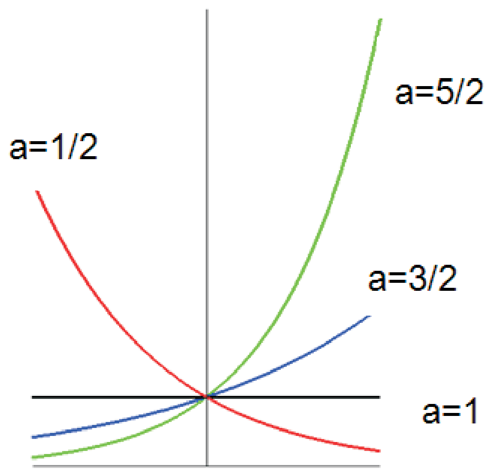


FIG. 30. Different exponential functions

In calculus this expression is expressed using limit notation as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e.$$

Exponent properties

(1) For n a positive integer: $a^n = a\cdot a\cdot\cdots\cdot a$ — n factors of a.
(2) For $n=0$: $a^0=1,\ a\neq 0$; 0^0 is not defined.
(3) For n a negative integer: $a^n=\frac{1}{a^{-n}},\ a\neq 0$.
(4) $a^m a^n = a^{m+n}$.
(5) $(a^n)^m = a^{mn}$.
(6) $(ab)^m = a^m b^m$.
(7) $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m},\ b\neq 0$.
(8) $\frac{a^m}{a^n}=a^{m-n}=\frac{1}{a^{n-m}},\ a\neq 0$.

Example: Solve $4^{x-2}=64$.
Solution:

$$4^{x-2}=64$$

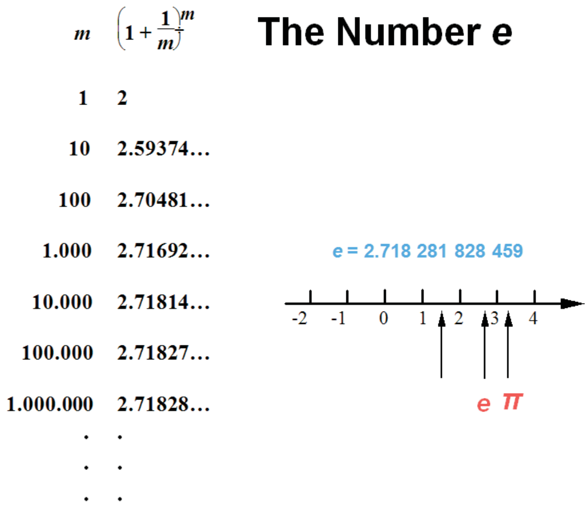


FIG. 31. Constant e

$$4^{x-2} = 4^3$$
$$x - 2 = 3$$
$$x = 5.$$

Answer: $x = 5$. ◇

Example: Solve $9^x + 3^{x+2} - 10 = 0$.
Solution:

$$9^x + 3^{x+2} - 10 = 0$$
$$(3^x)^2 + 3^2 3^x - 10 = 0$$
$$(3^x)^2 + 3^2 3^x - 10 = 0 \; - \; \textit{equation of quadratic type}$$
$$(3^x + 10)(3^x - 1) = 0$$
$$3^x = -10 \; or \; 3^x = 1$$
$$No\ solution\ or\ Solution\ x = 0.$$

Answer: $x = 0$. ◇

Example: Solve $3^{x+1} = 2^{2x-1}$.

Solution:

$$3^{x+1} = 2^{2x-1}$$
$$ln\ 3^{x+1} = ln\ 2^{2x-1}$$
$$(x+1)ln\ 3 = (2x-1)ln\ 2$$
$$xln\ 3 + ln\ 3 = 2xln\ 2 - ln\ 2$$
$$xln\ 3 - 2xln\ 2 = -ln\ 3 - ln\ 2$$
$$x(ln\ 3 - 2ln\ 2) = -(ln\ 3 + ln\ 2)$$
$$x = \frac{-(ln\ 3 + ln\ 2)}{ln\ 3 - 2ln\ 2} = -\frac{ln\ 6}{ln\ 3 - ln\ 4} = -\frac{ln\ 6}{ln(3/4)} \cong 6.23.$$

Answer: $x = -\frac{ln\ 6}{ln(3/4)} \cong 6.23.$ ◇

Exercise: Solve $3^{2x+1} = 81$.

Answer: $x = \frac{3}{2}$.

11.2. Problems

Solve equation.

11.1. $10^x = 0.00001$.
11.2. $\left(\frac{1}{8}\right)^x = 64$.
11.3. $7^x = \frac{1}{\sqrt[3]{49}}$.
11.4. $1^x = 12$.
11.5. $\left(\frac{2}{3}\right)^x = 1.5$.
11.6. $\left(\frac{2}{5}\right)^x = 6\frac{1}{4}$.
11.7. $12^x = 1$.
11.8. $3^x = \sqrt[3]{9}$.
11.9. $0.2^x = \sqrt{0.008}$.
11.10. $10^x = 4$.
11.11. $0.5^{x-2} = 0.25$.
11.12. $36^{-x} = 6$.
11.13. $6^{x^2-1} = 1$.
11.14. $16^{x-0.5} = 32^{14-x}$.
11.15. $5^x \times 2^x = 0.001$.
11.16. $(\sqrt{7})^{x-1}(\sqrt{2})^{x-1} = 1$.
11.17. $2^x + 2^{x+2} = 20$.
11.18. $4^{x-1} + 4^x + 4^{x+1} = 84$.

Solve inequality.

11.19. $5^x < 125$.
11.20. $4^x > 256$.
11.21. $10^x > 0.01$.
11.22. $2^x < \frac{1}{64}$.
11.23. $0.3^x < 0.09$.
11.24. $0.3^x > 11\frac{1}{9}$.
11.25. $\left(\frac{1}{5}\right)^x > \sqrt[3]{\frac{1}{25}}$.
11.26. $3^x > \frac{1}{\sqrt{3}}$.
11.27. $\left(\frac{2}{3}\right)^{3x} < \left(\frac{3}{2}\right)^5$.

11.28. $25^{-x} < 5\sqrt{5}$.

11.29. $(\sqrt{3})^x \times 3 > \frac{1}{27}$.

11.30. $27^x 3^{1-x} < \frac{1}{3}$.

12. Logarithmic Functions

DEFINITION 12.1. *The* **logarithmic function** *to the base a, where* $a > 0$ *and* $a \neq 1$, *is denoted by* $y = log_a x$ *and is defined by*

$$y = log_a x \text{ if and only if } x = a^y.$$

The domain of the logarithmic function $y = log_a x$, *is* $x > 0$.

<u>**Exercise:**</u> Change exponential expression into an equivalent logarithmic expression.

Answer:

$$a^7 = z \text{ means } log_a Z = 7$$
$$10^2 = 100 \text{ means } log_{10} 100 = 2.$$

<u>**Exercise:**</u> Change logarithmic expression into an equivalent exponential expression.

Answer:

$$y = log_3 7 \text{ means } 3^y = 7$$
$$7 = log_3 a \text{ means } 3^7 = a.$$

Domain of logarithmic function = Range of exponential function $=(0, \infty)$.

Range of logarithmic function = Domain of exponential function $= (-\infty, \infty)$.

Properties of the Graph of a Logarithmic Function

$$f(x) = log_a x$$

- The x-intercept of the graph is 1. There is no y-intercept.
- The y-axis is a vertical asymptote of the graph.
- A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.
- The graph is smooth and continuous, with no corners or gaps.

DEFINITION 12.2. **The Natural Logarithm** $y = ln\ x$ *if and only if* $x = e^y$.

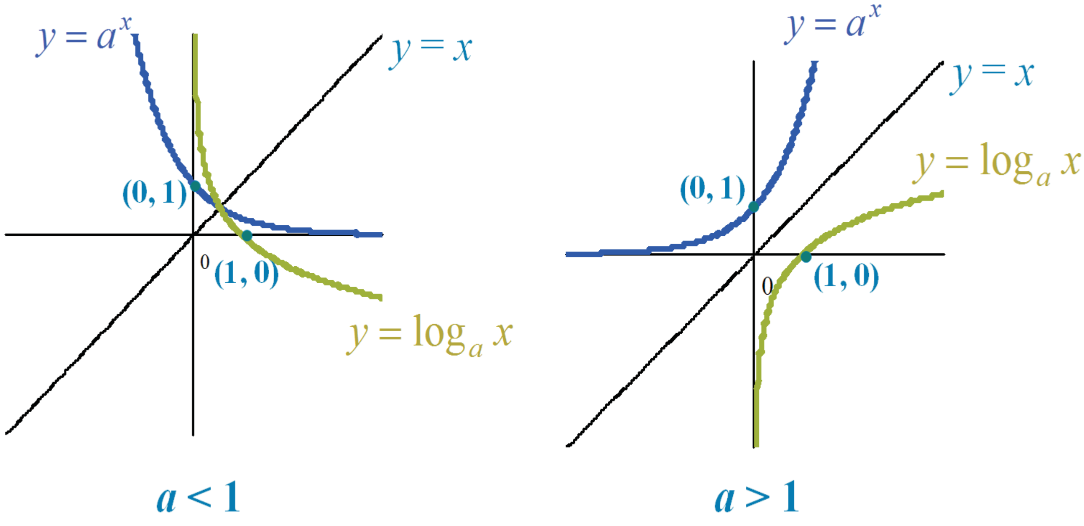
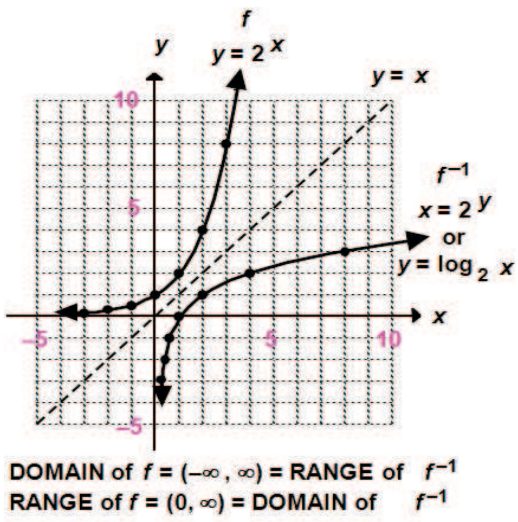


FIG. 32. Graph of the logarithmic function



f		f^{-1}	
x	$y = 2^x$	$x = 2^y$	y
-3	$\frac{1}{8}$	$\frac{1}{8}$	-3
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

FIG. 33. Logarithmic Function with Base 2

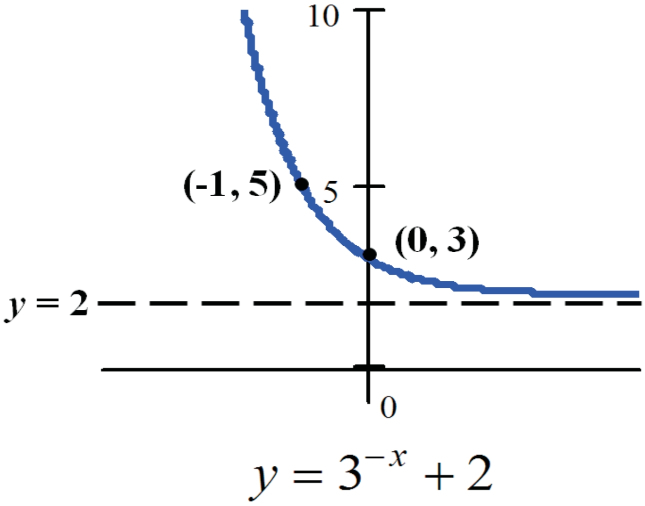


FIG. 34. Graph of the function $f(x) = ln\ (x-3)$

Exercise: Graph $f(x) = ln\ (x-3)$. Determinate the domain, range and vertical asymptote.

Answer:
Graph: see figure 34
Domain: $(3, \infty)$ or $x > 3$
Range: All reals
Vertical Asymptote: x = 3.

DEFINITION 12.3. **The Common Logarithmic Function (base=10)** *$y = lg\ x = log\ x$ if and only if $x = 10^y$.*

Properties of Logarithms

(1) $log_a 1 = 0$
(2) $log_a a = 1$
(3) $log_a a^r = r$
(4) $a^{log_a r} = r$
(5) $log_a(bc) = log_a b + log_a c$
(6) $log_a \frac{b}{c} = log_a b - log_a c$

(7) $log_a b^c = c log_a b$

Exercise: Write the expression $log_a \frac{\sqrt{xy}}{z^3(y-1)}$ as sum and (or) difference of logarithms. Express all powers as factors.

Answer: $log_a \frac{\sqrt{xy}}{z^3(y-1)} = log_a \sqrt{xy} - log_a(z^3(y-1)) = \frac{1}{2} log_a(xy) - (log_a z^3 + log_a(y-1)) = \frac{1}{2}(log_a x + log_a y) - 3 log_a z - log_a(y-1) = \frac{1}{2} log_a x + \frac{1}{2} log_a y - 3 log_a z - log_a(y-1)$.

Answer: $log_a \frac{\sqrt{xy}}{z^3(y-1)} = \frac{1}{2} log_a x + \frac{1}{2} log_a y - 3 log_a z - log_a(y-1)$.

Exercise: Write the expression $3ln\ x - 2ln\ (y+1) - 0.5ln\ z$ as a single logarithm.

Answer: $3ln\ x - 2ln\ (y+1) - 0.5ln\ z = 3ln\ x - 2ln\ (y+1) - 0.5ln\ z = ln\ x^3 - ln\ (y+1)^2 - ln\ z^{0.5} = ln \frac{x^3}{(y+1)^2} - ln\sqrt{z} = ln \frac{x^3}{(y+1)^2\sqrt{z}}$.

Answer: $3ln\ x - 2ln\ (y+1) - 0.5ln\ z = ln \frac{x^3}{(y+1)^2\sqrt{z}}$.

Change-of-Base Formula

If $a \neq 1,\ b \neq 1$ and c are positive real numbers, then

$$log_a c = \frac{log_b c}{log_b a}$$

For computation we use:

$$log_a c = \frac{log\ c}{log\ a} = \frac{ln\ c}{ln\ a}$$

Example: $log_7 34 = \frac{log 34}{log 7} \approx 1.81$. ◇

In the following properties $a,\ b,\ c$ are positive real numbers, with $a \neq 1$.

$$If\ b = c,\ then\ log_a b = log_a c.$$

$$If\ log_a b = log_a c,\ then\ b = c.$$

12.1. Logarithmic and Exponential Equations

<u>**Exercise:**</u> Solve $log_3(2x+5) = 2$.

Answer:

$$log_3(2x+5) = 2$$

Rewrite in exponential form:

$$3^2 = 2x+5$$
$$9 = 2x+5$$
$$4 = 2x$$
$$x = 2.$$

Answer: $x = 2$.

<u>**Exercise:**</u> Solve $log_4(3x-5) = 2$.

Answer:

$$log_4(3x-2) = 2$$

Rewrite in exponential form:

$$4^2 = 3x-2$$
$$16 = 3x-2$$
$$18 = 3x$$
$$x = 6.$$

Answer: $x = 6$.

<u>**Exercise:**</u> Solve $log_6(x+3) + log_6(x-2) = 1$.

Answer:

$$log_6(x+3) + log_6(x-2) = 1$$
$$log_6[(x+3)(x-2)] = 1$$
$$log_6(x^2+x-6) = 1$$
$$x^2+x-6 = 6$$
$$x^2+x-12 = 0$$
$$(x+3)(x-3) = 0.$$
$$x = -4 \; or \; x = 3.$$

Check!
Check: $x = -4$
$log_6(-4+3) + log_6(-4-2) = log_6(-1) + log_6(-6)$

Both terms are undefined.
Check: $x = 3$
$log_6(3+3) + log_6(3-2) = log_6(6) + log_6(1) = 1$
Answer: $x = 3$.

12.2. Problems

Find the domain of functions.

12.1. $y = \lg(-x)$.
12.2. $y = \ln x^2$.
12.3. $\lg(|x| - x)$.
12.4. $10^{\frac{1}{\log_x 10}}$.

Solve equations.
12.5. $\lg(x + 1.5) = -\lg x$.
12.6. $\frac{\log_2(9-2^x)}{3-x} = 1$
12.7. $\log_4(x+3) - \log_4(x-1) = 2 - \log_4 8$.
12.8. $\log_5(x-2) + \log_{\sqrt{5}}(x-2) + \log_{0.2}(x-2) = 4$.
12.9. $\lg(5-x) - \frac{1}{3}\lg(35 - x^3) = 0$.
Solve inequalities.
12.10. $\log_5(3x-1) < 1$.
12.11. $\log_{0.2}(4-2x) > -1$.
12.12. $log_{0.4}(2x-5) > \log_{0.4}(x+1)$.
12.13. $log_4(3x-1) < log_4(2x+3)$.
12.14. $log_{\frac{1}{2}}(x^2 - 5x + 6) > -1$.
12.15. $log_3 \frac{2-2x}{x} \geq -1$.
12.16. $log_3(x+2)(x+4) + log_{\frac{1}{3}}(x+2) < \frac{1}{2} log_{\sqrt{3}} 7$.
12.17. $log_{x-1} \frac{1}{2} > \frac{1}{2}$.

CHAPTER 4

TRIGONOMETRY

1. Angles and Their Measure

A **ray**, or half-line, is that portion of a line that starts at a point V on the line and extends indefinitely in one direction. The starting point V of a ray is called its vertex. If two lines are drawn with a common vertex, they form an **angle**. One of the rays of an angle is called the **initial side** and the other the **terminal side** (see fig. 1).

An angle θ is said to be in **standard position** if its vertex is in the origin of a rectangular coordinate system and the initial side coincides with the x-axis (see fig. 1).

When an angle θ is in standard position, the terminal side either will lie in a quadrant, in which case we say θ lies in that quadrant, or it will lie on the x-axis or the y-axis, in which case we say θ is a quadrant angle.

Angles are commonly measured in either

- degrees
- radians

The angle formed by rotating the initial side exactly once in the counterclockwise direction until it coincides with itself (1 revolution) is said to measure 360 degrees, abbreviated 360° (see fig. 2). One degree, 1°, is $\frac{1}{360}$ revolution.

A **right angle** is an angle of 90° or $\frac{1}{4}$ revolution. A **straight angle** is an angle of 180° or $\frac{1}{2}$ revolution.

One minute, denoted $1'$, is defined as $\frac{1}{60}$ degree. One second, denoted $1''$, is defined as $\frac{1}{60}$ minute.

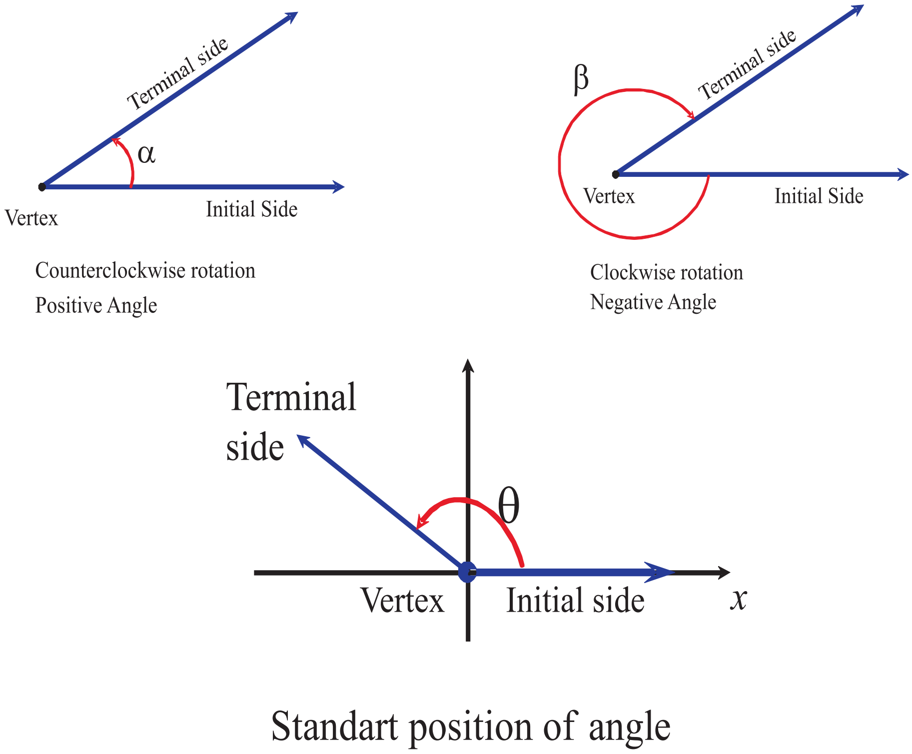


FIG. 1. Positive angle, negative angle, angle in standard position

Example: Convert $30°12'55''$ to a decimal in degrees.
$30°12'55'' = 30° + 12 \times \frac{1}{60} + 55 \times \frac{1}{3600} = 30° + 0.2° + 0.01528° = 30,21528°$. ◇

Example: Convert $45.413°$ to $D°M'S''$ form.
$45.413° = 45° + 0.413° \frac{60'}{1°} = 45° + 24.78' = 45° + 24' + 0.78' \frac{60''}{1'} = 45° + 24' + 46.8'' = 45°24'47''$. ◇

Consider a circle of radius r. Construct an angle whose vertex is at the center of this circle, called the central angle, and whose rays

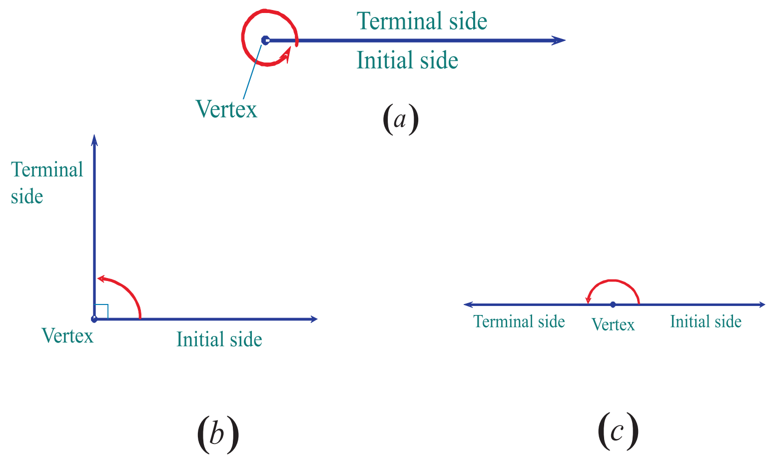


FIG. 2. Angles of (a) 360° or 2π radians, (b) 90° or $\frac{\pi}{2}$ radians, (c) 180° or π radians

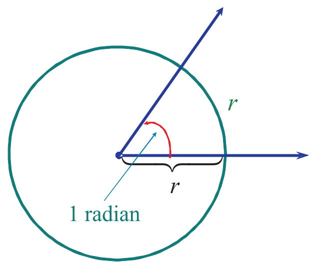


FIG. 3. Angle of 1 radian

subtend an arc on the circle whose length is r (see fig. 3). The measure of such an angle is **1 radian**.

For a circle of radius r, a central angle of θ radians subtends an arc whose length is $r\theta$.

Then $360° = 2\pi$ radians, $180° = \pi$ radians, $1° = \frac{\pi}{180}$ radians and $1\ \textit{radian} = \left(\frac{180}{\pi}\right)^{\circ}$.

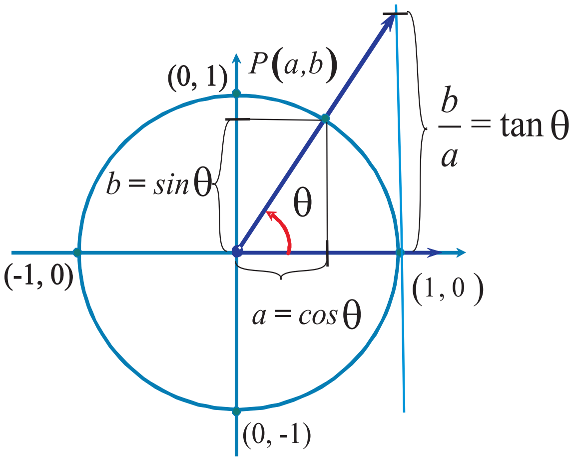


FIG. 4. Definition of trigonometric functions

Example: Convert 135° to radians.
$135° = 135° \frac{\pi}{180°} = \frac{3\pi}{4}$ radians. ◇

Example: Convert $-\frac{2\pi}{3}$ to degrees.
$-\frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{180°}{\pi} = -120°$. ◇

2. Trigonometric Function

The *unit circle* is a circle whose radius is 1 and whose center is at the origin.

Let θ be an angle in right position and $P = (a, b)$ be the point of intersection of the terminal side of t and the unit circle (see fig. 4).

The **cosine function** associates with θ is the x-coordinate of P and is denoted by

$$a = \cos\theta$$

. The **sine function** associates with θ is the y-coordinate of P and is denoted by

$$b = \sin\theta$$

.

If $a \neq 0$, then the **tangent function** is

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}.$$

From considering a right triangle with vertex at the points $O = (0,0)$, $P = (a,b)$ and $(a,0)$ and applying the Pythagorean theorem to its sides, one can get the **Pythagorean Identity**:

$$\sin^2\theta + \cos^2\theta = 1.$$

Example: Find $\cos\theta, tan\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$.

Solution: by Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$, then $\cos^2\theta = 1 - \sin^2\theta = 1 - \left(\frac{2}{5}\right)^2 = \frac{9}{25}$. So $\cos\theta$ equals either $\frac{3}{5}$ or $-\frac{3}{5}$. Since $\frac{\pi}{2} < \theta < \pi$ we assume that $\cos\theta = -\frac{3}{5}$. Now $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{2}{3}$. $\diamond$

Exercise: Find $\tan\theta$ if $cos\theta = \frac{24}{25}$ and θ is in Quadrant 3.

Answer: $-\frac{7}{24}$.

In additional to sine, cosine and tangent function we will use three functions defined below.
Cosecant: $\csc\theta = \frac{1}{\sin\theta}$;
secant: $\sec\theta = \frac{1}{\cos\theta}$;
cotangent: $\cot\theta = \frac{1}{\tan\theta}$.

2.1. Trigonometrical Functions of Right Triangle

Given a right triangle with one of the angles named θ in standard position, and the sides of the triangle relative to θ named opposite, adjacent, and hypotenuse (see fig. 5), we define the six trigonometrical functions to be:

(1) **sine:** $\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r}$;
(2) **cosine:** $\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r}$;
(3) **tangent:** $\tan\theta = \frac{opposite}{adjacent} = \frac{y}{x}$;
(4) **cosecant:** $\csc\theta = \frac{1}{sin\theta} = \frac{r}{y}$;
(5) **secant:** $\sec\theta = \frac{1}{cos\theta} = \frac{r}{x}$;
(6) **cotangent:** $\cot\theta = \frac{1}{tan\theta} = \frac{x}{y}$.

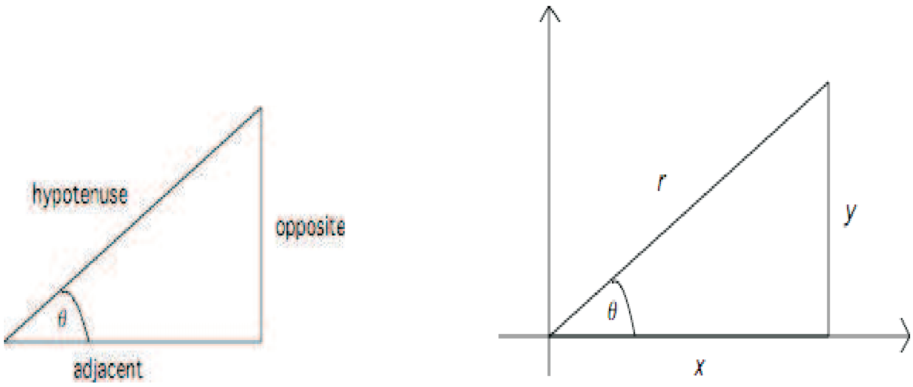


FIG. 5. Defining of trigonometrical function of right triangle

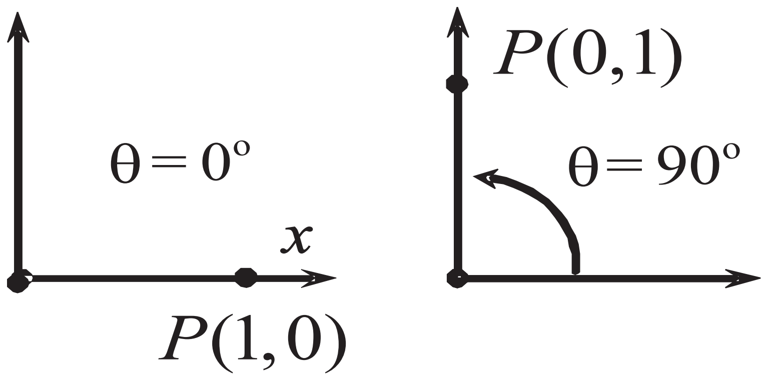


FIG. 6. Finding of trigonometrical function for 0° and 90°

Exercise: Find the six trigonometric function of an angle θ of right triangle on the fig. 5, if sides of the triangle are 3 and 2.

2.2. Trigonometric Function of Quadrant Angles

In this section we find the trigonometrical function of quadrant angles. Let's start from 0° and 90°. Look at the picture in fig. 6 and apply the definition of trig function to get the following result.

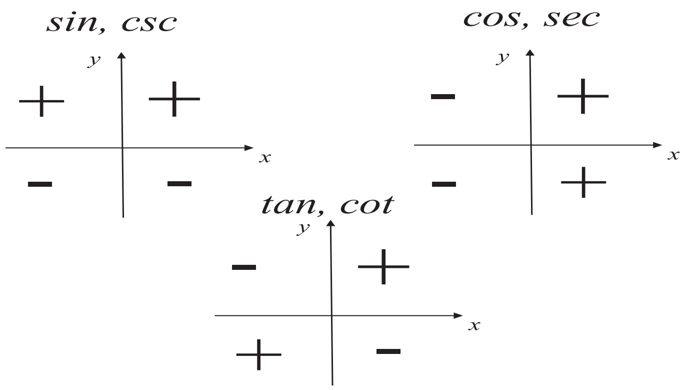


FIG. 7. Signs of trig functions

$\theta°$	θ, radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	0	0	1	0	undef.	1	undef.
90	$\frac{\pi}{2}$	1	0	undef.	1	undef.	0

Similarly one can get trig functions for remainder quadrant angles.

$\theta°$	θ, radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
180	π	0	-1	0	undef.	-1	undef.
270	$\frac{3\pi}{2}$	-1	0	undef.	-1	undef.	0

Exercise: Find the values of trig functions for 2π, $-90°$, 35π.

2.3. Sign of Trigonometric Function

The signs of trig functions are given in fig. 7. One can get it immediately from the definition of the functions.

2.4. Finding of Trig Function for Base Angles

From geometry we know the relationships of sides in the special right triangles shewn in the fig. 8. It gives a possibility to find values of trig function for several basic angles that are presented in the following table.

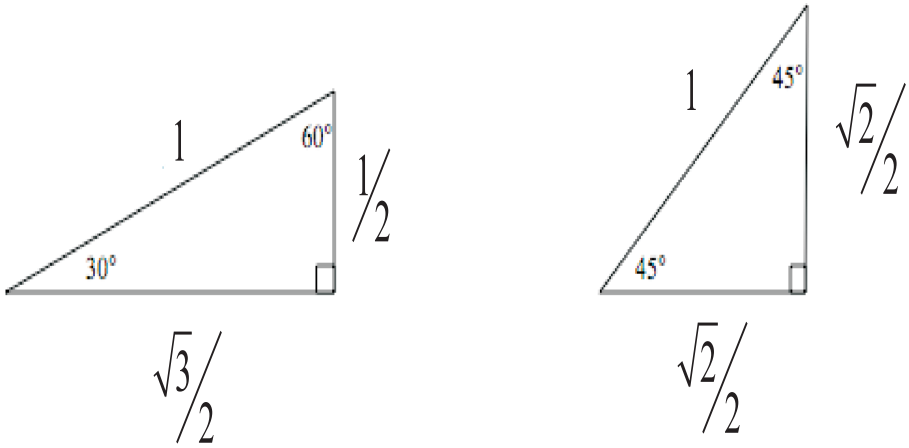


FIG. 8. Special triangles

$\theta°$	θ, radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{sqrt3}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
60	$\frac{\pi}{3}$	$\frac{sqrt3}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1

Exercise: Find values of

(1) $\sin\frac{2\pi}{3}$,
(2) $\cos\frac{7\pi}{6}$,
(3) $\tan\frac{3\pi}{4}$.

Answer: (1) $\frac{\sqrt{3}}{2}$, (2) $-\frac{\sqrt{3}}{2}$, (3) -1.

2.5. Period of Trig Function

A function f is called **periodic** if there is a positive number p such that whenever θ is in the domain of f, so is $\theta + p$, and

$$f(\theta + p) = f(\theta).$$

If there is the smallest such number p, its value is called the **(fundamental) period** of f.

It follows from the definition that the trig function are periodic and satisfy the conditions below.

$$\begin{array}{ll} \sin(\theta + 2\pi) = \sin(\theta); & \csc(\theta + 2\pi) = \csc(\theta); \\ \cos(\theta + 2\pi) = \cos(\theta); & \sec(\theta + 2\pi) = \sec(\theta); \\ \tan(\theta + \pi) = \tan(\theta); & \cot(\theta + \pi) = \cot(\theta). \end{array}$$

Example: Find the value of $\sin(390°)$.
$\sin(390°) = \sin(360° + 30°) = \sin(30°) = \frac{1}{2}$. ◇

<u>Exercise:</u> Find the value of $\cot(\frac{13\pi}{4})$.

Answer: 1.

2.6. Even-Odd Properties of Trig Functions

Use definitions to verify the following properties of trig functions.

$$\begin{array}{ll} \sin(-\theta) = -\sin(\theta); & \csc(-\theta) = -\csc(\theta); \\ \cos(-\theta) = \cos(\theta); & \sec(-\theta) = \sec(\theta); \\ \tan(-\theta) = -\tan(\theta); & \cot(-\theta) = -\cot(\theta). \end{array}$$

Example: Find the value of $\sin(-30°)$.
$\sin(-30°) = -\sin(30°) = -\frac{1}{2}$. ◇

Example: Find the value of $\cot(\frac{7\pi}{4})$.
$\cot(\frac{7\pi}{4}) = \cot(\frac{(8-1)\pi}{4}) = \cot(2\pi - \frac{\pi}{4}) = \cot(-\frac{\pi}{4}) = -\cot(\frac{7\pi}{4}) = -1$. ◇

2.7. Problems

Find the value of following trig functions.

2.1. $\sin \frac{11\pi}{6}$.

2.2. $\cos \frac{4\pi}{3}$.

2.3. $\sec\frac{5\pi}{4}$.

2.4. $\csc\frac{7\pi}{4}$.

2.5. $\tan\frac{4\pi}{3}$.

2.6. $\cos\frac{5\pi}{3}$.

2.7. $\cot\frac{5\pi}{6}$.

2.8. $\sin\frac{29\pi}{6}$.

2.9. $\cos\frac{35\pi}{3}$.

2.10. $\tan\frac{23\pi}{3}$.

2.8. Answers

2.1.$-\frac{1}{2}$. **2.2.**$-\frac{1}{2}$. **2.3.**$-\sqrt{2}$. **2.4.**$-\sqrt{2}$. **2.5.**$\sqrt{3}$. **2.6.**$\frac{1}{2}$. **2.7.**$-\sqrt{3}$. **2.8.**$\frac{1}{2}$. **2.9.**$\frac{\sqrt{3}}{2}$. **2.10.**$-\sqrt{3}$.

3. Graphs of Trigonometrical Functions. The Simplest Equation and Inequalities

Graphs of six trigonometrical functions are given in the fig. 9-11.

Example: Use graph of $\sin x$ to solve the equation $\sin x = 0$. Solution: list x-intercepts of $\sin x$ to obtain $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. The general form of all that values is $x = \pi k$, where k is any integer.
Answer: $x = \pi k$, $k \in \mathbb{Z}$. ◇

Example: Use graph of $\cos x$ to solve the inequality $\cos x > 0$ for $x \in [-\pi, \pi]$.
Solution: We should find all values of x from the segment $[-\pi, \pi]$ for which the graph is above the $x-$ axis. They are $-\frac{\pi}{2} < x < \frac{\pi}{2}$. ◇

Example: Use graph of $\cos x$ to solve the inequality $\cos x > 0$ for all real.
Solution: We should find all values of x for which the graph is above the $x-$ axis. Since $y = \cos(x)$ is periodic with period 2π we should add $2\pi k$ to the result of the previous exercise.
Answer: $-\frac{\pi}{2} + 2\pi k < x < \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$. ◇

<u>**Exercise:**</u> Find all solution of equation $cosx = 0$.

Answer: $x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$.

Exercise: Find all solution of inequality $tanx \geq 0$.

Answer: $\pi k \leq x < \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$.

4. Basic Trigonometrical Identities

This section is devoted to review of trigonometrical identities and their application for simplifying expressions.

4.1. Pythagorean Identity

Let's remain the Pythagorean Identity connecting sin and cosine functions:

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

If we divide both side of this identity by $cos^2\theta$ or by $sin^2\theta$, we will get the following useful identities.

$$\boxed{1 + \tan^2\theta = \sec^2\theta} \qquad \boxed{1 + \cot^2\theta = \csc^2\theta}$$

Example: Symplify $\tan^2\theta - \frac{1}{\cos^2\theta}$.
Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$ we get $\tan^2\theta - \frac{1}{\cos^2\theta} = \frac{\sin^2\theta}{\cos^2\theta} - -\frac{1}{\cos^2\theta} = \frac{\sin^2\theta - 1}{\cos^2\theta}$.
It follows from the Pythagorean Identity that $\sin^2\theta - 1 = -\cos^2\theta$. So we finally get $\frac{\sin^2\theta - 1}{\cos^2\theta} = \frac{-\cos^2\theta}{\cos^2\theta} = -1$. ◇

Exercise: Symplify $\cot^2\theta - \frac{1}{\sin^2\theta}$.

Exercise: Sumplify $(\sec\theta - \cot\theta)(\sec\theta + \cot\theta)$.

Answer: 1.

4.2. Sum and Difference Formulas

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}}$$

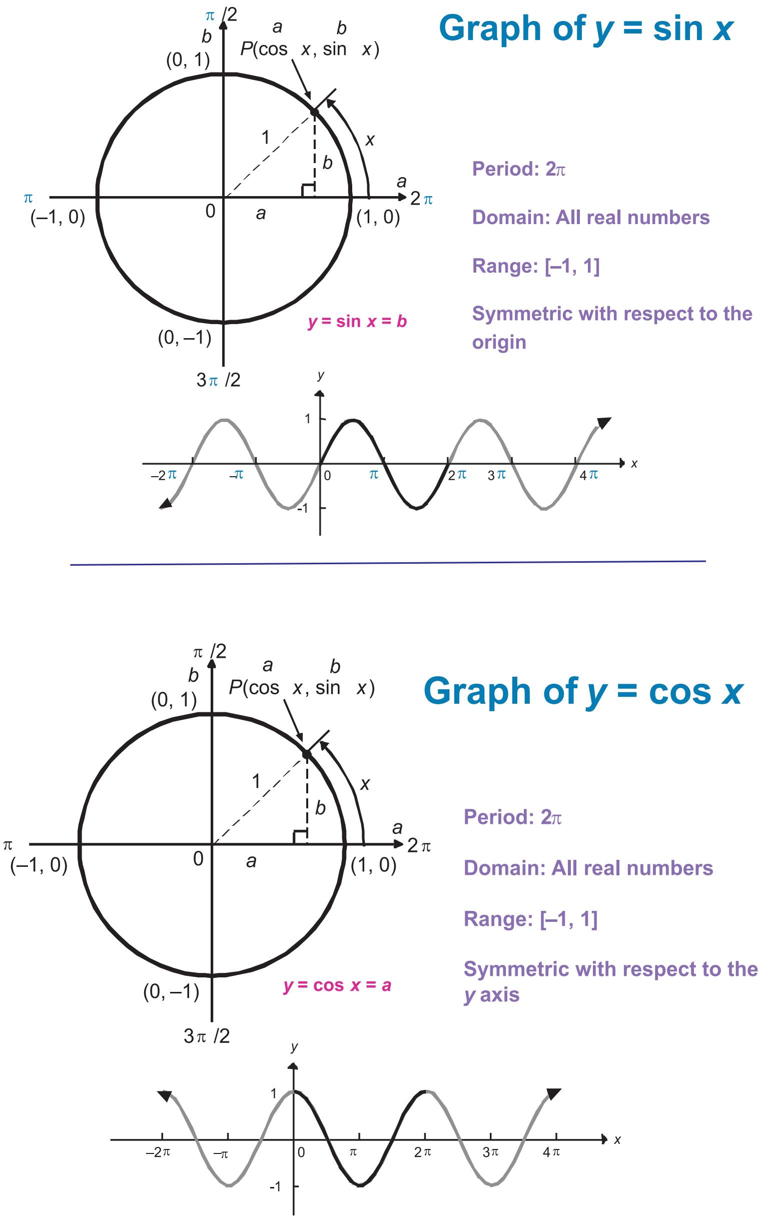


FIG. 9. Graphs of functions $y = sinx$ and $y = cosx$

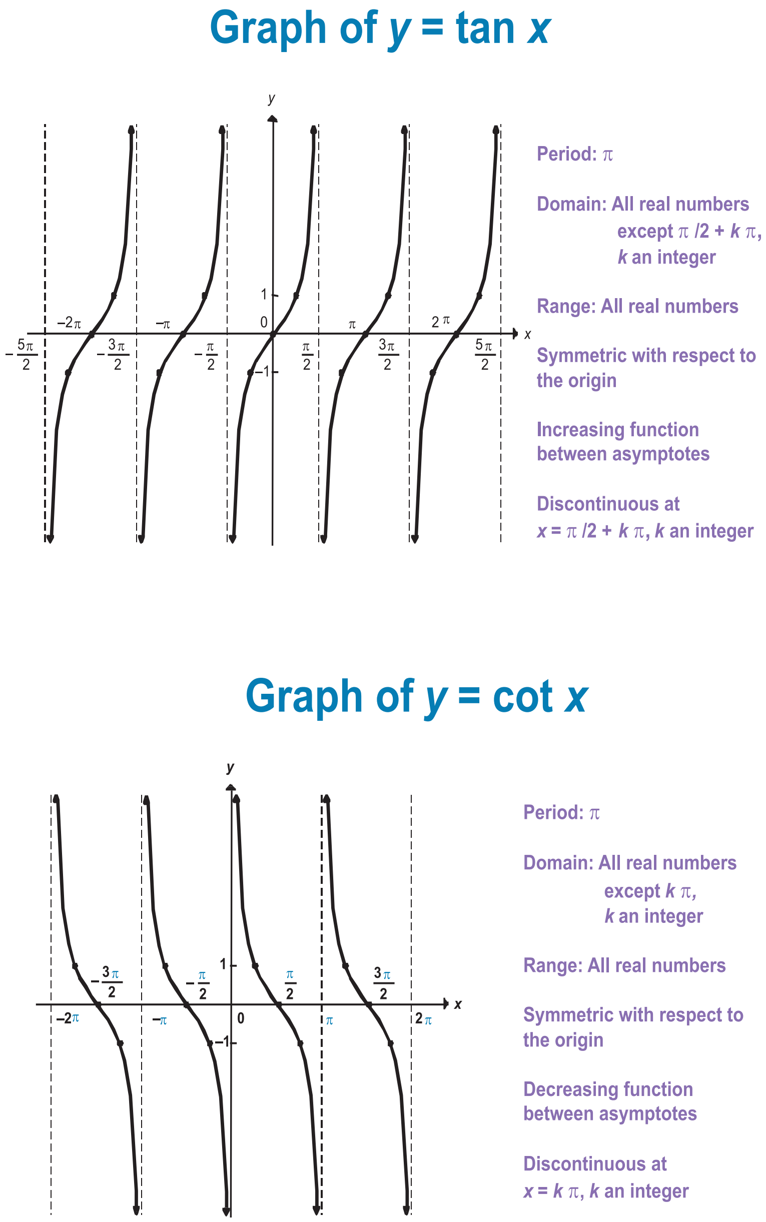


FIG. 10. Graphs of functions $y = tanx$ and $y = cotx$

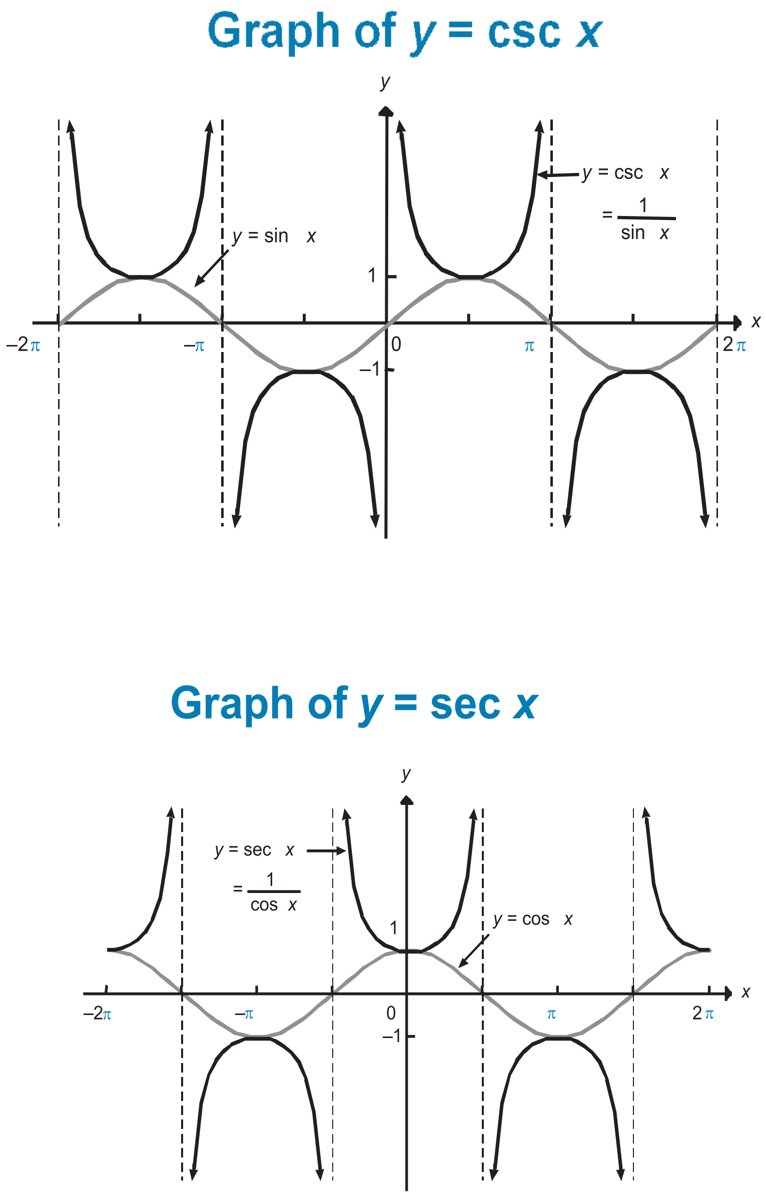


FIG. 11. Graphs of functions $y = cscx$ and $y = secx$

Example: Find the value of $\sin(-\frac{\pi}{12})$.

$\sin(-\frac{\pi}{12}) = \sin(\frac{\pi}{4} - \frac{\pi}{3}) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} - \cos\alpha\sin\frac{\pi}{3} = \frac{\sqrt{2}}{2} \times \frac{1}{2} - \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} = \frac{\sqrt{2}-\sqrt{6}}{4}$. ◇

<u>Exercise:</u> Find the value of $\cos(75°)$.

Answer: $\frac{\sqrt{6}-\sqrt{2}}{4}$.

<u>Exercise:</u> Symplify

(1) $\sin(\theta + \frac{\pi}{2})$.
(2) $\cos(\pi - \theta)$.

Answer: (1) $\cos\theta$. (2) $-\cos\theta$.

$$\boxed{\begin{array}{l} \tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ \\ \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \end{array}}$$

<u>Exercise:</u> Find $\cot(105°)$.

Answer: $\frac{1-\sqrt{3}}{1+\sqrt{3}}$.

<u>Exercise:</u> Symplify $\tan(\theta + \frac{\pi}{2})$; $\cot(\frac{3\pi}{2} - \theta)$.

Answer: $-\cot\theta$; $\tan\theta$.

4.3. Double-Angle Formulas

$$\boxed{\begin{array}{lcl} \cos 2\theta &=& \cos^2\theta - \sin^2\theta \\ \sin 2\theta &=& 2\sin\theta\cos\theta \\ \tan 2\theta &=& \frac{2\tan\theta}{1-\tan^2\theta} \end{array}}$$

Example: Find $\sin 2\theta$, $\cos 2\theta$ if $\sin\theta = \frac{5}{13}$ and $\frac{\pi}{2} < \theta < \pi$.

Solution: By Pythagorean Identity $\cos^2\theta + \sin^\theta = 1$, then $\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{25}{169} = \frac{144}{169}$. Since $\frac{\pi}{2} < \theta < \pi$, $\cos\theta$ must be negative, so $\cos\theta = -\sqrt{\frac{144}{169}} = -\frac{12}{13}$.

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{5}{13}\left(-\frac{12}{13}\right) = -\frac{120}{169}$;
$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\frac{12}{13}\right)^2 - \left(\frac{5}{13}\right)^2 = \frac{119}{169}$. ◇

Exercise: Find $\sin 6\theta$ if $\sin 3\theta = \sqrt{7}4$ and $\frac{\pi}{6} < \theta < \frac{pi}{3}$.

Answer:$-\frac{3\sqrt{7}}{8}$.

4.4. Half-angle Formulas

$$\boxed{\cos^2\theta = \tfrac{1+\cos 2\theta}{2};\ \sin^2\theta = \tfrac{1-\cos 2\theta}{2}}$$

Exercise: Deduce half-angle formula for $\tan^2\theta$.

Exercise: Use half-angle formulas to rewrite $\sin^2\theta\cos^2\theta$ without using powers of trig functions.

4.5. Product-to-Sum Formulas

$$\boxed{\begin{array}{l} \sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \\ \cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \\ \sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \end{array}}$$

Exercise: Express each product as a sum containing only sines and cosines: (1) $\sin 3\theta\sin 6\theta$; (2) $\sin 3\theta\cos 6\theta$

4.6. Establishing the Identity

The direction 'establish the identity' means to show, through the use of basic identities and algebraic manipulation, that one side of an equation is the same as the other side of the equation.

Example: Establish the identity: $1 - \frac{1}{2}\sin 2\theta = \frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$.

Solution: $\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta} = \frac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta} = \sin^2\theta - \sin\theta\cos\theta + \cos^2\theta = \sin^2\theta + \cos^2\theta - \sin\theta\cos\theta = 1 - \sin\theta\cos\theta = 1 - \frac{1}{2}\sin 2\theta$. ◇

Exercise: Establish the identity: $\frac{\cos 73^\circ\cos 13^\circ - \cos 17^\circ\cos 77^\circ}{\cos 123^\circ\cos 37^\circ - \cos 33^\circ\cos 53^\circ} = 1$

4.7. Problems

4.1. Find $\csc\theta$ if $\tan\theta = \frac{5}{12}$ and θ is in Quadrant 3.
4.2. Find $\sin\theta$ if $\cos\theta = \frac{12}{13}$ and $\frac{3\pi}{2} < \theta < 2\pi$.
4.3. Find $\sec\theta$ if $\tan\theta = \frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$.
4.4. Find $\cot\theta$ if $\sec\theta = \frac{25}{7}$ and $\frac{3\pi}{2} < \theta < 2\pi$.
4.5. Simplify $\sqrt{\frac{1}{\sin^2\theta} - 1}$ if θ is in Quadrant 2.
4.6. Simplify $\sqrt{9 - 9\sin^2\theta}$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
4.7. Find $\cos\frac{\theta}{2}$ if $\sin\theta = -\frac{5}{8}$ and $270° < \theta < 360°$.
4.8. Find $\sin 4\theta$ if $\sin\theta = \frac{4}{5}$ and $\theta \in (\frac{pi}{2}, \pi)$.
4.9. Rewrite $\cos^4\theta$ without using powers of trig functions.
Simplify the expression.
4.10. $\cos^4\theta - \sin^2\theta$.
4.11. $\frac{\tan\theta + \sin\theta}{2\cos^2\frac{\theta}{2}}$.
4.12. $\frac{\cos 4\theta + 1}{\cot\theta - \tan\theta}$.
Establish the identity.
4.13. $\frac{\sin^4\theta + 2\sin\theta\cos\theta - \cos^4\theta}{\tan 2\theta - 1} = \cos 2\theta$.
4.14. $\frac{\tan 2\theta \tan\theta}{\tan 2\theta - \tan\theta} = \sin 2\theta$.
4.15. $3 - 4\cos 2\theta + \cos 4\theta = 8\sin^4\theta$.

4.8. Answers

4.1.$-\frac{13}{5}$. **4.2.**$-\frac{25}{13}$. **4.3.**$-\frac{5}{4}$. **4.4.**$-\frac{7}{24}$. **4.5.**$-\cot\theta$. **4.6.**$3\cos\theta$.

5. Trig Equations. Inverse Trigonometrical Functions

In this section are presented examples of trig equations that require of application of different tools like substitution, factorising and using trig identities.

Example: Solve the equation $\sin\theta = \frac{\sqrt{3}}{2}$.

To solve the equation look at the fig. 12 and remember the definition of sine. One can see two angles — $\theta = \frac{\pi}{3}$ and $\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ — that have sine value of $\frac{\sqrt{3}}{2}$, so that the angles satisfy the equation. Since sine is periodic function of period 2π, the angles $\frac{\pi}{3} + 2\pi k$, $\frac{2\pi}{3} + 2\pi k$ also satisfy the equation for any integer k.

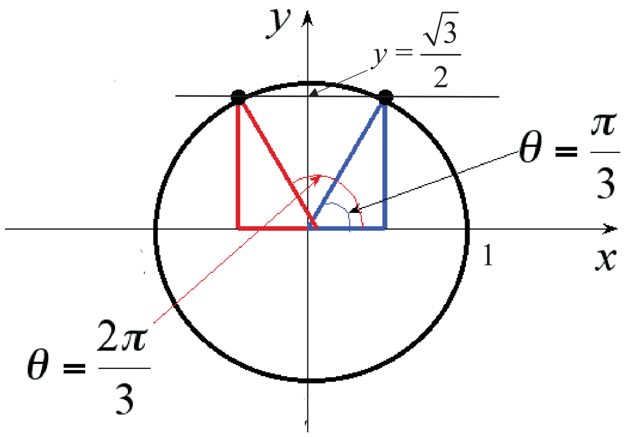


FIG. 12. Solutions of the equation $\sin\theta = \frac{\sqrt{3}}{2}$

Answer: $\theta = \frac{\pi}{3} + 2\pi k$ or $\frac{2\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$. ◇

Example: Solve the equation $2\cos^2\theta + \cos\theta - 1 = 0$, $0 \leq \theta < 2\pi$. Put $t = \cos\theta$. Then the initial equation is reduced to

$$2t^2 + t - 1 = 0.$$

Factorising brings

$$(2t-1)(t+1) = 0$$

and, so,

$$2t - 1 = 0 \text{ or } t + 1 = 0.$$

Then we have

$$t = \frac{1}{2} \text{ or } t = -1;$$
$$\cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1.$$

Solution of last equations are shewn in the fig. 13. There are $\theta = \frac{\pi}{3}$, $\theta = -\frac{\pi}{3}$ and $\theta = \pi$. ◇

Example: Solve the equation $\frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta = 1$ on $0 \leq \theta < 2\pi$.

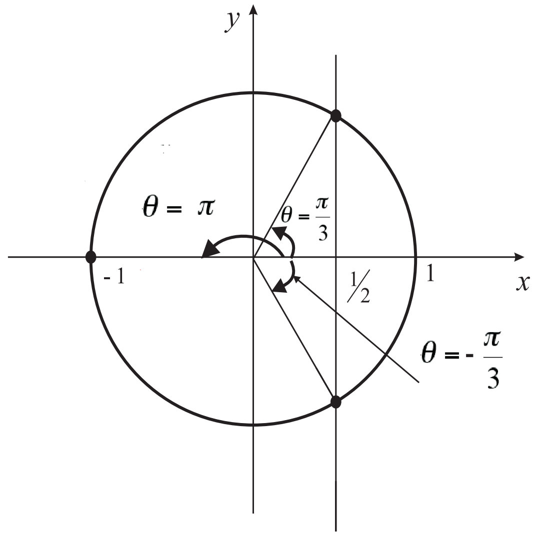


FIG. 13. Solutions of the equations $\cos\theta = \frac{1}{2}$ and $\cos\theta = -1$

Solution: Denote that $\frac{1}{2} = \sin\frac{\pi}{6}$, $\frac{\sqrt{3}}{2} = cos\frac{\pi}{6}$. Then we can rewrite the equation as

$$\sin\frac{\pi}{6}\sin\theta + \cos\frac{\pi}{6}\cos\theta = 1.$$

Use an addition formula to get

$$\cos(\theta - \frac{\pi}{6}) = 1.$$

Draw the picture similarly to one at the previous example or look at the graph of cosine function to get

$$\theta - \frac{\pi}{6} = 0$$

.

Finally, $\theta = \frac{\pi}{6}$. ◇

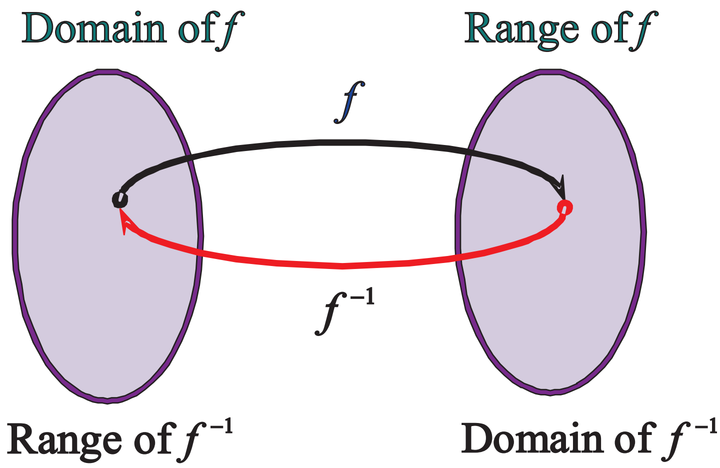


FIG. 14. Definition of Inverse Function

5.1. Inverse Function

If a function $f : A \to B$ is injective, then one can solve x in terms of y from the equation $y = f(x)$ provided that y is in the range of f. This defines the inverse function of the function f (see fig. 14).

DEFINITION 5.1. *Let $f : A \to B$ be a function. If there is a function $g : B \to A$ such that $f \circ g$ is the identity on B and $g \circ f$ is the identity on A, then the function f is called* **invertible** *and the function g is called the* **inverse function** *of the function f.*

The inverse function g is denoted by f^{-1}. Do not confuse $f^{-1}(x)$ with $f(x)^{-1} = \frac{1}{f(x)}$!

5.2. Inverse Trig Function

Trig functions are not injective. Then to define an inverse function for any trig function, it is necessary to consider a restriction of the function on the subset of its domain such that the restriction is injective.

To define $\sin^{-1}$ (we will call it 'arcsine') we consider the function $y = \sin x$ restricted on the set $-\frac{\pi}{2} < x < \frac{\pi}{2}$ (see fig. 16).

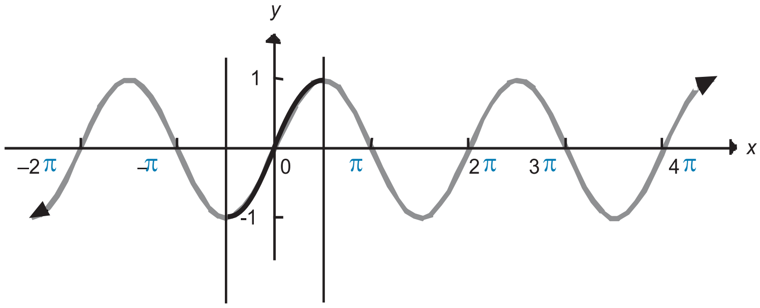


FIG. 15. $y = \sin x$

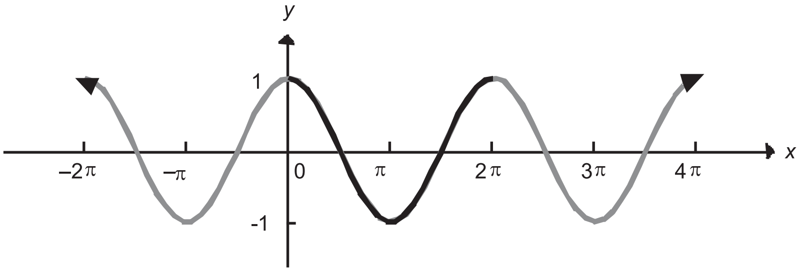


FIG. 16. $y = \cos x$

DEFINITION 5.2. *For any $x \in [-1, 1]$ the function $y = \sin^{-1}(x) = \arcsin x$ is the function which returns the angle $y \in [-\frac{\pi}{2}, \frac{pi}{2}]$ such that $\sin y = x$.*

To define $\cos^{-1}$ ('arccosine') we consider the function $y = \cos x$ restricted on the set $0 < x < \pi$ (see fig. 16).

DEFINITION 5.3. *For any $x \in [-1, 1]$ the function $y = \cos^{-1}(x) = \arccos x$ is the function which returns the angle $y \in [0, \pi]$ such that $\cos y = x$.*

DEFINITION 5.4. *For any real x the function $y = \tan^{-1}(x) = \arctan x$ is the function which returns the angle $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that $\tan y = x$.*

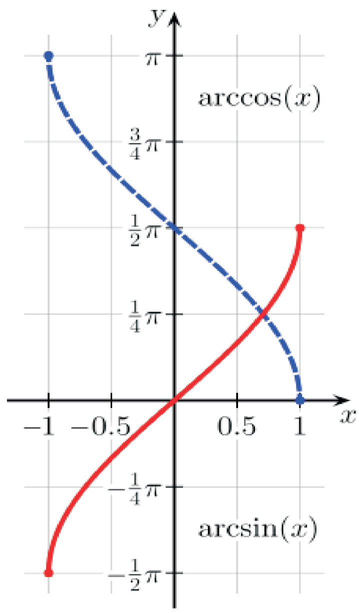


FIG. 17. Graphs of functions $y = \arcsin x$, $y = \arccos x$

DEFINITION 5.5. *For any real x the function $y = \cot^{-1}(x) = \operatorname{arccot} x$ is the function which returns the angle $y \in [0, \pi]$ such that $\cot y = x$.*

See graphs of inverse trig functions in the fig. 17, 18.

Example: Find $\arcsin \frac{1}{2}$.

Solution: we should find an angle θ such that $\sin\theta = \frac{1}{2}$ and $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. That is $\theta = \frac{\pi}{6}$.

Answer: $\arcsin \frac{1}{2} = \frac{\pi}{6}$. ◇

Exercise: Find value of

(1) $\arcsin(-\frac{1}{2})$;
(2) $\arccos(\frac{1}{2})$;
(3) $\arctan \sqrt{3}$;
(4) $\operatorname{arccot} 1$.

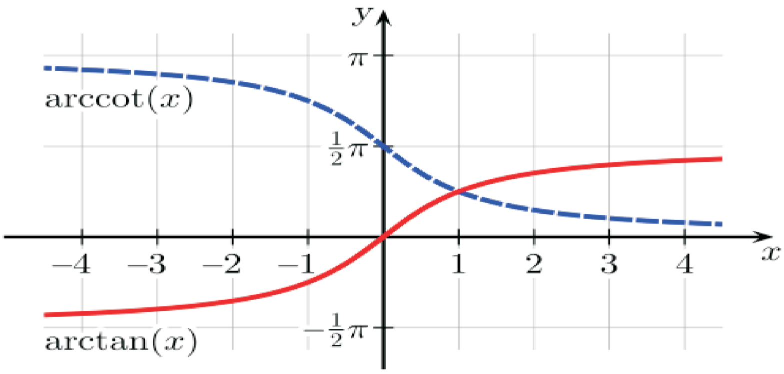


FIG. 18. Graphs of functions $y = \arctan x$, $y = \operatorname{arccot} x$

Answer: (1) $-\frac{\pi}{6}$; (2) $\frac{\pi}{3}$; (3) $\frac{\pi}{3}$; (4) $\frac{\pi}{4}$.

5.3. General Formulas for Solution of the Simplest Trigonometric Equations

Here are given the general formulas for solution of simplest trigonometric equations. Everywhere below k is any integer number.

equation	solution	alternative notation
$\sin x = a$	$\begin{cases} x = \arcsin a + 2\pi k \text{ or} \\ x = \pi - \arcsin a + 2\pi k \end{cases}$	$x = (-1)^{-1} \arcsin a + \pi k$
$\cos x = a$	$\begin{cases} x = \arccos a + 2\pi k \text{ or} \\ x = -\arccos a + 2\pi k \end{cases}$	$x = \pm \arccos a + 2\pi k$
$\tan x = a$	$x = \arctan a + \pi k$	
$\cot x = a$	$x = \operatorname{arccot}\ a + \pi k$	

Exercise: Draw the picture illustrating the general formulas.

Exercise: Use the general formulas to solve equations.

(1) $\sin x = -\frac{1}{2}$;
(2) $\cos x = \frac{1}{2}$;
(3) $\tan x = \sqrt{3}$;

(4) $\cot x = 1$.

Answer:

(1) $x = (-1)^k \arcsin\left(-\frac{1}{2}\right) + \pi k = (-1)^k\left(-\frac{\pi}{6}\right) + \pi k = (-1)^{k+1}\frac{\pi}{6} + \pi k$, $k \in \mathbb{Z}$;
(2) $x = \pm \arccos\left(\frac{1}{2}\right) + 2\pi k = \pm\frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$;
(3) $x = \arctan\sqrt{3} + \pi k = \frac{\pi}{3} + \pi k$, $k \in \mathbb{Z}$;
(4) $x = \operatorname{arccot} 1 + \pi k = \frac{\pi}{4} + \pi k$, $k \in \mathbb{Z}$.

5.4. Problems

Solve equations.
5.1. $\sin^2\theta + \sin\theta = 0$ for $0 \le \theta < 2\pi$.
5.2. $\cos^2\theta = 1$.
5.3. $\sin\theta = \sqrt{3}\cos\theta$ for $0 \le \theta < 2\pi$.
5.4. $\sin 2\theta = \cos\theta$, $0 \le \theta < 2\pi$.
5.5. $2\sin^2\theta - \sin\theta = 0$, $0 \le \theta < 2\pi$.
5.6. $\cos\theta - \sin\theta = 1$, $0 \le \theta < 2\pi$.
5.7. $\tan^2\theta = \tan\theta$.
5.8. $2\sin^2\theta - \cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

5.5. Answers

5.1.$\theta \in \{0, \pi, \frac{3\pi}{2}\}$. **5.2.**$\theta = \pi k$, $k \in \mathbb{Z}$. **5.3.**$\theta \in \{\frac{\pi}{3}, \frac{4\pi}{3}\}$. **5.4.**$\theta \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$. **5.5.**$\theta \in \{0, \pi, \frac{\pi}{2}\}$. **5.6.**$\theta \in \{0, \frac{3\pi}{2}\}$. **5.7.**$\theta = \frac{\pi}{4} + \pi k$, $k \in \mathbb{Z}$. **5.8.**$\theta \in \{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\}$.

Question Cards for Examination

This book covers material of 1st semester. Detailed programme of the course is given by table of contents. Here we append approximate question cards for examination.

Card 1
(1) Number Systems: Natural numbers, Integers, Rational numbers.
(2) Polynomial Functions. Linear Polynomial functions.
(3) Reduce the fraction $\frac{\frac{3}{7}}{1.2}$.
(4) Calculate $\frac{10.5-4.5\cdot 3.5}{4\frac{6}{11}\cdot(1\frac{1}{5}-1\frac{1}{2})}$.
(5) Simplify the expression $6\sqrt{2\frac{1}{3}}-\sqrt{84}+4\sqrt{1\frac{5}{16}}$.
(6) Simplify the expression $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}-\frac{a^2+b^2}{a^2-b^2}\right)\cdot\frac{2a^2-2b^2}{a^2+b^2}$.
(7) Solve the inequality $|x+6|>-5$.
(8) Find slope for linear function $3y+6=0$, graph it.
(9) Solve the inequality $x^2-9x+20<0$ and graph the function $y=x^2-9x+20$.
(10) Solve the inequality $x^4+x^3-20x^2\leq 0$ and graph the function $y=x^4+x^3-20x^2$.
(11) Simplify the expression $\sqrt{x^2+4}$ for $x=2tg\theta$ with $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

Card 2
(1) Rules of actions with fractions.
(2) Piecewise Defined Functions.
(3) Reduce the fraction $\frac{540}{960}$.
(4) Convert 78% to a fraction.
(5) Simplify the expression $\left(125x^{-6}\right)^{\frac{2}{3}}$.
(6) Calculate $\sqrt[5]{6+2\sqrt{17}}\cdot\sqrt[5]{6-2\sqrt{17}}$.

(7) Solve the inequality $|x+6| \leq -2$.
(8) Graph the function $f(x) = |x+2|$ and find its values at the points $4.2, -0.5, 8$.
(9) Solve the inequality $\frac{x}{x-4} < \frac{x-5}{x+1}$ and graph the function $y = \frac{x}{x-4} - \frac{x-5}{x+1}$.
(10) Use the graph of $y = sinx$ to sketch the graph of $y = 2 + 2sin(x - \frac{\pi}{2})$.
(11) Solve the equation $cosec^2 2x = 2$.

Card 3
(1) Converting of fractions.
(2) Power functions. Simple Deformations.
(3) Reduce the fraction $\frac{540}{960}$.
(4) Convert 0.65 to percentages.
(5) Simplify the expression $\left(81a^{-8}\right)^{-\frac{3}{4}}$.
(6) Simplify the expression $\frac{4!+6!}{6!-3!}$.
(7) Rewrite the equation of the following circle in canonical form and use it to find the centre and the radius of the circle: $x^2 - 6x + y^2 - 4y = -4$. Also, graph this circle.
(8) Graph the function $f(x) = |x-3|$ and find its values at the points $4.2, -0.5, 8$.
(9) Solve the inequality $x \geq \frac{16}{x}$ and graph the function $y = x - \frac{16}{x}$.
(10) Use the graph of $y = cosx$ to sketch the graph of $y = -1 - 3cos(x+\pi)$.
(11) Solve the equation $\left(\frac{1}{9}\right)^{2x-3} = 9 \cdot 3^{6x-10}$.

Card 4
(1) Irrational numbers: Algebraic numbers, Transcendental numbers. Real numbers.
(2) Quadratic function. Quadratic Formula.
(3) Write the decimal 3.12(05) as the quotient of two natural numbers.
(4) Convert $-0.6(7)$ to the quotient.
(5) Simplify the expression $\frac{\sqrt[6]{x\sqrt[3]{x}}}{x^{-\frac{7}{9}}}$.
(6) Simplify the expression $\frac{(n+1)!+(n+3)!}{n!-(n+2)!}$.

(7) Rewrite the equation of the following circle in canonical form and use it to find the centre and the radius of the circle: $2x^2 - 6x + 2y^2 - 4y = 0$. Also, graph this circle.
(8) Graph the function $f(x) = |x|$ and find its value: $|4.2|, |-0.5|, |8|$.
(9) Solve the inequality $\frac{2x^3-6x}{(x^2-1)^3} > 0$ and graph the function $y = \frac{2x^3-6x}{(x^2-1)^3}$.
(10) Use the graph of $y = ctgx$ to sketch the graph of $y = 1 + ctg(x - \pi)$.
(11) Solve the equation $0.2 \cdot 5^{2-x} = 25^{x-1}$.

Card 5 (1) Definition and Properties of Exponent, Rational Exponent.
(2) Rational Inequalities.
(3) Write the decimal 0.3(22) as the quotient of two natural numbers.
(4) Convert $-5.8(45)$ to the quotient.
(5) Simplify the expression $\frac{\sqrt[5]{a^2\sqrt[4]{a^{-3}}}}{a^{-\frac{3}{4}}}$.
(6) Calculate C_{10}^5.
(7) Rewrite the equation of the following circle in canonical form and use it to find the centre and the radius of the circle: $x^2 - x + y^2 + y = 1$. Also, graph this circle.
(8) Draw the graph of the function $y = |x-1| + |x+3|$.
(9) Solve the inequality $\frac{x^2(2x^2-5x+3)}{(x-4)^2(3x-1)(x+2)} \leq 0$ and graph the function $y = \frac{x^2(2x^2-5x+3)}{(x-4)^2(3x-1)(x+2)}$.
(10) Use the graph of $y = tgx$ to sketch the graph of $y = -2tg(x - \frac{\pi}{2})$.
(11) Solve the equation $\left(2\frac{1}{3}\right)^{x+4} = \left(\frac{3}{7}\right)^{x^2+1}$.

Card 6 (1) Special Formulas.
(2) Angles and Their Measure. Trigonometric Functions.
(3) Write the fraction $\frac{7}{16}$ as a decimal.
(4) Convert $-\frac{85}{9}$ to the decimal.
(5) Calculate $(\sqrt[3]{5} + \sqrt[3]{2})(\sqrt[3]{25} - \sqrt[3]{10} + \sqrt[3]{4})$.
(6) Calculate C_{k+3}^k.

(7) Rewrite the equation of the following circle in canonical form and use it to find the centre and the radius of the circle: $3x^2-6x+3y^2-12y=-3$. Also, graph this circle.
(8) Draw the graph of the function $y = x - |x+2| + 3$.
(9) Convert 150^0 to radians.
(10) Use an formula to simplify $sin(\frac{\pi}{2} - x)$.
(11) Solve the equation $0.2 \cdot 5^{x^2-x} = 125^{x-1}$.

Card 7 (1) The Binomial Theorem.
(2) Pythagorean Identities. Table for the values of the trigonometric functions.
(3) Write the decimal -4.22 as the quotient of two positive integers.
(4) Convert -31.48 to the quotient.
(5) Calculate $\left(7\sqrt{\frac{5}{7}} - 5\sqrt{\frac{7}{5}}\right)^2$.
(6) Expand $(2x-3)^3$ using the Binomial Theorem.
(7) Rewrite the equation of the following circle in canonical form and use it to find the centre and the radius of the circle: $y^2 + 2y + x^2 - 2x = 0$. Also, graph this circle.
(8) Draw the graph of the function $y = |x+2| - 2|x-3|$.
(9) Convert $\frac{2\pi}{5}$ to degrees.
(10) Use an formula to simplify $cos(\frac{x}{2} + \pi)$.
(11) Solve the equation $lnx = 2ln3 - 2$.

Card 8 (1) Absolute value.
(2) Graphs of Trigonometric Functions.
(3) Write the fraction $-\frac{33}{4}$ as a decimal.
(4) Convert 30% to a fraction.
(5) Simplify the expression $2\frac{2}{3}x^2y^8 \cdot \left(-1\frac{1}{2}xy^3\right)^4$.
(6) Expand $(a+2b)^4$ using the Binomial Theorem.
(7) Find the natural domain of the function $f(x) = \sqrt{2x^2+3x+1}$.
(8) Draw the graph of the function $y = 3 - (x+2)^2$.
(9) Calculate by definition $tg150^0$.
(10) Use formulas to simplify $(sinx + cosx)^2$.
(11) Solve the equation $lg(3-5x) = \frac{1}{2}lg36 + lg2$.

Card 9
(1) Power function.
(2) Sum and Difference Formulas. Double-Angle and Half-Angle Formulas.
(3) Calculate $\frac{0.15-0.15\cdot 6.4}{-\frac{3}{8}+0.175}$.
(4) Convert 1.44 to percentages.
(5) Simplify the expression $\left(2\frac{1}{3}a^4b^8\right)^3 \cdot \left(-1\frac{2}{7}a^5b^{12}\right)$.
(6) Expand $(n-2)^4+(n+2)^4$ using the Binomial Theorem.
(7) Find the natural domain of the function $f(x) = \ln(-2x^2+3x-1)$.
(8) Draw the graph of the function $y = -1+(x-1)^3$.
(9) Calculate by definition $ctg(-\frac{2\pi}{3})$.
(10) Use an formula to simplify $tg(3x-\frac{3\pi}{2})$.
(11) Solve the equation $log_2(x^2+3x+3) = 0.5log_2 4 + log_2 0.5$.

Card 10
(1) Concept of a function. Graphs of Functions. Symmetry. Monotonous Functions.
(2) The simplest Trigonometric Equations. Inverse Functions.
(3) Calculate $\frac{3.2\cdot 1.5-6.3}{1\frac{2}{15}\cdot(1\frac{1}{2}-\frac{2}{5})}$.
(4) Prove that $\sqrt[4]{\sqrt{3}+5}$ is algebraic.
(5) Simplify the expression $a^{2n+5} : (a^n)^2$.
(6) Find all real solutions to $||x+2|-3| = 3$.
(7) Find the natural domain of the function $f(x) = \sqrt[4]{\frac{x^2-5x+4}{x^2-5x+6}}$.
(8) Draw the graph of the function $y = 2-(x-2)^4$.
(9) Calculate by definition $sec(-45^0)$.
(10) Use an formula to simplify $ctg(\pi-2x)$.
(11) Solve the equation $2^x = 9$.

Card 11
(1) Converting of fractions.
(2) Exponential function. The Number e.
(3) Calculate $\frac{-12.5\cdot 2.4+23\frac{2}{5}}{6\frac{5}{12}-1\frac{3}{8}\cdot 2}$.
(4) Prove that $\sqrt[5]{\sqrt[3]{2}-1}$ is algebraic.
(5) Simplify the expression $x^{n-2}\cdot x^{3-n}\cdot x$.
(6) Find all real solutions to $|2x-1|+|x+2| = 2$.

(7) Find the natural domain of the function $f(x) = \ln \frac{-x^2+5x-4}{2x^2+5x+8}$.
(8) Draw the graph of the function $y = (x+1)^5 - 2$.
(9) Calculate by definition $cosec(-\frac{3\pi}{4})$.
(10) Find $cos2x$ if $cosx = \frac{4}{5}$ and $0 < x < \frac{\pi}{2}$.
(11) Solve the inequality $\left(\frac{2}{3}\right)^{2-x^2} < \left(\frac{9}{4}\right)^{2x+1}$.

Card 12 (1) Rules of actions with fractions.
(2) Logarithmic Functions. The Natural Logarithm. Properties of Logarithmic Functions.
(3) Calculate $\frac{(-8.03:1.1+3.9)\cdot\frac{5}{16}}{(\frac{1}{8}-\frac{3}{4})\cdot 68}$.
(4) Prove that $\sqrt[6]{\sqrt[4]{2}+2}$ is algebraic.
(5) Simplify the expression $\left(a + \frac{b^2}{a-b}\right)\left(1 - \frac{b^3}{a^3+b^3}\right)(a+b)$.
(6) Find all real solutions to $|-x+1| < 3$.
(7) Find the natural domain of the function $f(x) = \sqrt[3]{\frac{1}{2x^2+5x+8}}$.
(8) Find the vertex of the parabola $y = 2x^2 - 8x + 7$ and graph it.
(9) Find $\cos x$ if $tgx = -5$ and $\frac{3\pi}{2} < x < 2\pi$.
(10) Find tgx if $cos2x = -\frac{3}{5}$ and $\frac{\pi}{2} < x < \frac{3\pi}{4}$.
(11) Solve the inequality $4^x - 4\cdot 2^x + 3 \geq 0$.

Card 13 (1) Converting of fractions.
(2) Logarithmic and Exponential Equations and Inequalities.
(3) Calculate $\frac{(0.11\cdot 8.6-5.946):0.025}{1\frac{2}{9}-1\frac{5}{18}\cdot 36}$.
(4) Calculate $-(-2.5)^{-1}\cdot(-2.5)^2-(16^0)^{\frac{1}{2}}-(0.2)^{-2}+125^{\frac{1}{3}}\cdot 0.2$.
(5) Simplify the expression $\frac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}} - \frac{a+b}{a^{\frac{1}{3}}+b^{\frac{1}{3}}}$.
(6) Find all real solutions to $x^2 + |x-2| < 1$.
(7) Find x-intercept and y-intercept for linear function $2x+3y+6=0$, graph it.
(8) Factorize quadratic expression $y = 2x^2 - 8x + 7$ and graph it.

(9) Find $\sin x$ if $tgx = \frac{7}{4}$ and $\pi < x < \frac{3\pi}{2}$.
(10) Solve the equation $2cos^2x - sinx = 1$.
(11) Solve the inequality $lnx^2 + 4lnx + ln1 > 0$.

Card 14
(1) Irrational numbers: Algebraic numbers, Transcendental numbers. Real numbers.
(2) Sum and Difference Formulas. Double-Angle and Half-Angle Formulas.
(3) Calculate $\frac{-0.032\cdot 10.25+0.118}{1:(5\frac{2}{7}-2\frac{8}{21}\cdot 2)}$.
(4) Calculate $(-1.5)^{-3} - (\frac{2}{5})^{-4} \cdot (\frac{2}{5})^3 - \left((\frac{4}{9})^{0.8}\right)^0 + 16^{\frac{3}{4}} \cdot 0.5$.
(5) Simplify the expression $\frac{a^3-b^3}{a+b} : \left(a + \frac{b^2}{b+a}\right) - \frac{b^2}{b+a} \cdot \frac{a^2-b^2}{b^2+ab}$.
(6) Find all real solutions to $|-x+1| < 3$.
(7) Find x-intercept and y-intercept for linear function $12x - 3y + 24 = 0$, graph it.
(8) Factorize quadratic expression $y = -x^2 - 2x - 1$.
(9) Find tgx if $cosecx = -\frac{5}{4}$ and $\frac{\pi}{2} < x < \pi$.
(10) Solve the equation $3sin2x = -1$.
(11) Solve the inequality $log_{\frac{1}{2}}(x^2 + 3x + 3) \leq 8$.

Card 15
(1) Special Formulas.
(2) Angles and Their Measure. Trigonometric Functions.
(3) Calculate $\frac{6.25\cdot 4.8-23.4}{(2\frac{3}{4}\cdot 2-12\frac{5}{6}):5}$.
(4) Simplify the expression $10\sqrt{\frac{2}{5}} - 0.5\sqrt{160} + 3\sqrt{1\frac{1}{9}}$.
(5) Simplify the expression $\left(\frac{x+2}{x+1} - \frac{8x^2-8}{x^3-1} : \frac{4x+4}{x^2+x+1}\right) : \frac{1}{x+1}$.
(6) Solve the inequality $|x^2 - 9| \geq 4$.
(7) Find slope for linear function $x - 3y + 2 = 0$, graph it.
(8) Factorize quadratic expression $y = -2x^2 + 8$ and graph it.
(9) Simplify the expression $(sec4x - 1)(sec4x + 1)$.
(10) Solve the equation $tg^2x - 4 = 0$.
(11) Solve the equation $ctg^2x - 3ctgx + 2 = 0$.

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Введение в математический анализ

Учебно-методическое пособие

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